

The Inverse Laplace Transform

The inverse Laplace transform of the function $F(s)$ is the unique function $f(t)$ that is continuous on $[0, \infty)$ and satisfies $\mathcal{L}\{f(t)\} = F(s)$, we will use the notation

$\{\mathcal{L}^{-1}[F(s)]\}$ to denote the inverse Laplace transform of $F(s)$, and we will write:

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

Linearity of the Inverse Transform

let $f_1(t)$ and $f_2(t)$ be the inverse Laplace transforms of $F_1(s)$ and $F_2(s)$ respectively and let C be a constant we have:

$$\mathcal{L}^{-1}[CF_1(s)] = C\mathcal{L}^{-1}[F_1(s)] = Cf_1(t)$$

$$\mathcal{L}^{-1}[F_1(s) \mp F_2(s)] = \mathcal{L}^{-1}[F_1(s)] \mp \mathcal{L}^{-1}[F_2(s)] = f_1(t) \mp f_2(t)$$

$$\mathcal{L}^{-1}[C_1F_1(s) \mp C_2F_2(s)] = C_1\mathcal{L}^{-1}[F_1(s)] \mp C_2\mathcal{L}^{-1}[F_2(s)] = C_1f_1(t) \mp C_2f_2(t)$$

(1)

(2)

Examples

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$$

$$\textcircled{2} \mathcal{L}^{-1}\left[\frac{3}{s}\right] = 3\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 3$$

$$\boxed{\mathcal{L} t^n = \frac{n!}{s^{n+1}}}$$

$$\textcircled{3} \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$$

$$\textcircled{4} \mathcal{L}^{-1}\left[\frac{7}{s^3}\right] = \frac{7}{2!} \mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = \frac{7}{2} t^2$$

$$\begin{aligned} \textcircled{5} \mathcal{L}^{-1}\left[\frac{3s+4}{s^2+4}\right] &= \mathcal{L}^{-1}\frac{3s}{s^2+4} + \mathcal{L}^{-1}\frac{4}{s^2+4} = 3\mathcal{L}^{-1}\frac{s}{s^2+4} + 2\mathcal{L}^{-1}\frac{2}{s^2+4} \\ &= 3\cos 2t + 2\sin 2t \end{aligned}$$

$$\textcircled{6} \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$$

$$\begin{aligned} \textcircled{7} \mathcal{L}^{-1}\left\{\frac{2}{(s-5)^4}\right\} &= 2\mathcal{L}^{-1}\left\{\frac{1}{(s-5)^4}\right\} = 2e^{-5t}\mathcal{L}^{-1}\frac{1}{s^4} = \frac{2}{3!}e^{-5t}\mathcal{L}^{-1}\frac{3!}{s^4} \\ &= \frac{1}{3}e^{-5t}t^3 \end{aligned}$$

$$\textcircled{8} \mathcal{L}^{-1}\left\{\frac{6}{2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s-\frac{3}{2}}\right\} = 3e^{\frac{3}{2}t}$$

$$\begin{aligned} \textcircled{9} \mathcal{L}^{-1}\left\{\frac{3+4s}{9s^2-16}\right\} &= \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{3}{s^2-\frac{16}{9}}\right\} + \frac{4}{9}\mathcal{L}^{-1}\left\{\frac{s}{s^2-\frac{16}{9}}\right\} \\ &= \frac{1}{4}\sinh \frac{4}{3}t + \frac{4}{9}\cosh \frac{4}{3}t \end{aligned}$$

$$8) \mathcal{L}^{-1} \left\{ \frac{6}{2s-3} \right\} = \mathcal{L}^{-1} \left\{ \frac{6}{2(s-\frac{3}{2})} \right\} = \mathcal{L}^{-1} \frac{3}{s-\frac{3}{2}} = 3 \mathcal{L}^{-1} \frac{1}{s-\frac{3}{2}} = 3e^{+\frac{3}{2}t}$$

$$9) \mathcal{L}^{-1} \left\{ \frac{3+4s}{9s^2-16} \right\} = \mathcal{L}^{-1} \left\{ \frac{3+4s}{9(s^2-\frac{16}{9})} \right\} = \frac{1}{9} \mathcal{L}^{-1} \frac{3+4s}{s^2-\frac{16}{9}}$$

$$= \frac{3}{9} \mathcal{L}^{-1} \frac{1}{s^2-\frac{16}{9}} + \frac{4}{9} \mathcal{L}^{-1} \frac{s}{s^2-\frac{16}{9}}$$

$$= \frac{3}{9} \mathcal{L}^{-1} \frac{\frac{4}{3}}{s^2-\frac{16}{9}} + \frac{4}{9} \mathcal{L}^{-1} \frac{s}{s^2-\frac{16}{9}} = \frac{3}{9} \cdot \frac{3}{4} \mathcal{L}^{-1} \frac{\frac{4}{3}}{s^2-\frac{16}{9}} + \frac{4}{9} \mathcal{L}^{-1} \frac{s}{s^2-\frac{16}{9}}$$

$$\frac{1}{4} \sinh \frac{4}{3}t + \frac{4}{9} \cosh \frac{4}{3}t$$

$$10) \mathcal{L}^{-1} \left\{ \frac{3s-5}{4(s^2-5+\frac{37}{4})} \right\} = \frac{1}{4} \mathcal{L}^{-1} \frac{3s-5}{s^2-5+\frac{37}{4}}$$

$$= \frac{1}{4} \mathcal{L}^{-1} \frac{3s-5}{(s^2-s+\frac{1}{4})+\frac{37}{4}-\frac{1}{4}} = \frac{1}{4} \mathcal{L}^{-1} \frac{3s-5}{(s-\frac{1}{2})^2+9}$$

اضافه و طرح
تحت مربع مقابل "s"
مقابل s - 1
 $(-\frac{1}{2})^2 = \frac{1}{4}$

$$\frac{1}{4} \mathcal{L}^{-1} \frac{3s}{(s-\frac{1}{2})^2+9} + \frac{1}{4} \mathcal{L}^{-1} \frac{-5}{(s-\frac{1}{2})^2+9} = \frac{3}{4} \mathcal{L}^{-1} \frac{s}{(s-\frac{1}{2})^2+9} - \frac{5}{4} \mathcal{L}^{-1} \frac{1}{(s-\frac{1}{2})^2+9}$$

$$\frac{3}{4} \mathcal{L}^{-1} \frac{(s-\frac{1}{2})+\frac{1}{2}}{(s-\frac{1}{2})^2+9} - \frac{5}{4} \mathcal{L}^{-1} \frac{1}{(s-\frac{1}{2})^2+9} \cdot \frac{3}{3}$$

$$\frac{3}{4} \mathcal{L}^{-1} \frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2+9} + \frac{3}{4} \mathcal{L}^{-1} \frac{\frac{1}{2}}{(s-\frac{1}{2})^2+9} - \frac{5}{12} \mathcal{L}^{-1} \frac{3}{(s-\frac{1}{2})^2+9}$$

$$\frac{3}{4} e^{\frac{1}{2}t} \mathcal{L}^{-1} \frac{s}{s^2+9} + \frac{1}{8} e^{\frac{1}{2}t} \mathcal{L}^{-1} \frac{3}{s^2+9} - \frac{5}{12} e^{\frac{1}{2}t} \mathcal{L}^{-1} \frac{3}{s^2+9}$$

$$\frac{3}{4} e^{\frac{1}{2}t} \cos(3t) + \frac{1}{8} e^{\frac{1}{2}t} \sin(3t) - \frac{5}{12} e^{\frac{1}{2}t} \sin(3t)$$

$$\frac{3}{4} e^{\frac{1}{2}t} \cos(3t) - \frac{7}{24} e^{\frac{1}{2}t} \sin(3t)$$

$$(15) \mathcal{L}^{-1} \left\{ \frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9} \right\}$$

$$\mathcal{L}^{-1} \left\{ \frac{6}{2(s-\frac{3}{2})} - \frac{3+4s}{9(s^2-\frac{16}{9})} + \frac{8-6s}{16(s^2+\frac{9}{16})} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{3}{s-\frac{3}{2}} - \frac{3}{9(s^2-\frac{16}{9})} - \frac{4s}{9(s^2-\frac{16}{9})} + \frac{8}{16(s^2+\frac{9}{16})} - \frac{6s}{16(s^2+\frac{9}{16})} \right\}$$

$$= 3 \mathcal{L}^{-1} \frac{1}{s-\frac{3}{2}} - \frac{1}{3} \mathcal{L}^{-1} \frac{1}{s^2-\frac{16}{9}} - \frac{4}{9} \mathcal{L}^{-1} \frac{s}{s^2-\frac{16}{9}} + \frac{1}{2} \mathcal{L}^{-1} \frac{1}{s^2+\frac{9}{16}} - \frac{3}{8} \mathcal{L}^{-1} \frac{s}{s^2+\frac{9}{16}}$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{3} \mathcal{L}^{-1} \frac{\frac{4}{3}}{s^2-\frac{16}{9}} + \frac{4}{9} \cosh(\frac{4}{3}t) + \frac{1}{2} \mathcal{L}^{-1} \frac{\frac{3}{4}}{s^2+\frac{9}{16}} - \frac{3}{8} \cos(\frac{3}{4}t)$$

$$= 3e^{\frac{3}{2}t} - \frac{1}{4} \sinh(\frac{4}{3}t) - \frac{4}{9} \cosh(\frac{4}{3}t) + \frac{3}{8} \sinh(\frac{3}{4}t) - \frac{3}{8} \cos(\frac{3}{4}t)$$

(3)

$$\begin{aligned}
\left\{ \frac{3s-5}{4(s^2-s+\frac{37}{4})} \right\} &= \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{3s-5}{(s^2-s+\frac{1}{4})+9} \right\} = \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{3s-5}{(s-\frac{1}{2})^2+3^2} \right\} \\
&= \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{3s}{(s-\frac{1}{2})^2+3^2} \right\} - \frac{5}{4} \mathcal{L}^{-1} \left\{ \frac{1}{(s-\frac{1}{2})^2+3^2} \right\} = \\
&= \frac{3}{4} \mathcal{L}^{-1} \left\{ \frac{(s-\frac{1}{2})+\frac{1}{2}}{(s-\frac{1}{2})^2+9} \right\} - \frac{5}{12} \mathcal{L}^{-1} \left\{ \frac{3}{(s-\frac{1}{2})^2+3^2} \right\} \\
&= \frac{3}{4} \mathcal{L}^{-1} \left\{ \frac{(s-\frac{1}{2})}{(s-\frac{1}{2})^2+9} + \frac{3}{8} \frac{1}{(s-\frac{1}{2})^2+9} \right\} - \frac{5}{12} e^{\frac{1}{2}t} \sin 3t \\
&= \frac{3}{4} e^{\frac{1}{2}t} \cos 3t + \frac{1}{8} e^{\frac{1}{2}t} \sin 3t - \frac{5}{12} e^{\frac{1}{2}t} \sin 3t \\
&= \frac{3}{4} e^{\frac{1}{2}t} \cos 3t - \frac{7}{24} e^{\frac{1}{2}t} \sin 3t = \frac{1}{4} e^{\frac{1}{2}t} (3 \cos 3t - \frac{7}{6} \sin 3t) = f(t)
\end{aligned}$$

$$\begin{aligned}
(11) \quad \mathcal{L}^{-1} \left\{ \frac{3s-5}{4s^2+4s+1} \right\} &= \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{3s-5}{(s^2+s+\frac{1}{4})} \right\} = \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{3s-5}{(s+\frac{1}{2})^2} \right\} \\
&= \frac{3}{4} \mathcal{L}^{-1} \left\{ \frac{s}{(s+\frac{1}{2})^2} \right\} - \frac{5}{4} \mathcal{L}^{-1} \left\{ \frac{1}{(s+\frac{1}{2})^2} \right\} = \frac{3}{4} \mathcal{L}^{-1} \left\{ \frac{(s+\frac{1}{2})-\frac{1}{2}}{(s+\frac{1}{2})^2} \right\} - \frac{5}{4} \mathcal{L}^{-1} \left\{ \frac{1}{(s+\frac{1}{2})^2} \right\} \\
&= \frac{3}{4} \mathcal{L}^{-1} \left\{ \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2} \right\} - \frac{3}{8} \mathcal{L}^{-1} \left\{ \frac{1}{(s+\frac{1}{2})^2} \right\} - \frac{5}{4} \mathcal{L}^{-1} \left\{ \frac{1}{(s+\frac{1}{2})^2} \right\} \\
&= \frac{3}{4} \mathcal{L}^{-1} \left\{ \frac{1}{s+\frac{1}{2}} \right\} - \frac{13}{8} \mathcal{L}^{-1} \left\{ \frac{1}{(s+\frac{1}{2})^2} \right\} = \frac{3}{4} e^{-\frac{1}{2}t} - \frac{13}{8} e^{-\frac{1}{2}t} t \\
&= \frac{1}{4} e^{-\frac{1}{2}t} (3 - \frac{13}{2} t)
\end{aligned}$$

(4)

$$\begin{aligned} \frac{3}{s^2+4s+13} &= \mathcal{L}^{-1} \left\{ \frac{3}{(s^2+4s+4)-4+13} \right\} = \mathcal{L}^{-1} \frac{3}{(s+2)^2+9} \\ &= e^{-2t} \mathcal{L}^{-1} \frac{3}{s^2+9} = e^{-2t} \sin 3t \end{aligned}$$

$$(13) \mathcal{L}^{-1} \frac{1}{s(s^2+4)}$$

Solu: using $\int_0^t f(u) du = \frac{F(s)}{s}$

$$\therefore \mathcal{L}^{-1} \frac{1}{s(s^2+4)} = \mathcal{L}^{-1} \frac{F(s)}{s} \Rightarrow F(s) = \frac{1}{s^2+4}$$

$$\therefore \mathcal{L}^{-1} F(s) = f(u) \Rightarrow \mathcal{L}^{-1} \frac{1}{s^2+4} = \frac{1}{2} \mathcal{L}^{-1} \frac{2}{s^2+4} = \frac{1}{2} \sin 2u = f(u)$$

$$\begin{aligned} \therefore \mathcal{L}^{-1} F(s) = f(u) &\Rightarrow \int_0^t f(u) du = \int_0^t \frac{1}{2} \sin 2u du = -\frac{1}{4} \int_0^t 2 \sin 2u du \\ &= -\frac{1}{4} \cos 2u \Big|_0^t = \frac{1}{4} (1 - \cos 2t) \end{aligned}$$

$$\therefore \mathcal{L}^{-1} \frac{1}{s(s^2+4)} = \frac{1}{4} (1 - \cos 2t)$$

(14) Evaluate $\mathcal{L}^{-1} \left\{ \frac{1}{s^3(s^2+1)} \right\}$ since $\mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\} = \sin t$

we have $\mathcal{L}^{-1} \frac{F(s)}{s} = \int_0^t f(u) du$

$$\therefore \mathcal{L}^{-1} \frac{1}{s(s^2+1)} = \int_0^t \sin u du = 1 - \cos t$$

 $\Rightarrow$

(5)

$$\mathcal{L}^{-1} \frac{1}{s^2(s^2+1)} = \int_0^t (1-\cos u) du = t - \sin t$$

$$\text{and } \mathcal{L}^{-1} \frac{1}{s^3(s^2+1)} = \int_0^t (u - \sin u) du = \frac{t^2}{2} + \cos t - 1$$

$$f(t) = \frac{t^2}{2} + \cos t - 1$$

Note to check your solution take
 $\mathcal{L}\{f(t)\} = F(s)$ means:

$$\mathcal{L}\left\{\frac{t^2}{2} + \cos t - 1\right\} = \frac{1}{2} \frac{2!}{s^3} + \frac{s}{s^2+1} - \frac{1}{s} = \frac{1}{s^3} + \frac{s}{s^2+1} - \frac{1}{s}$$

$$F(s) = \frac{s^2+1+s^4-s^2(s^2+1)}{s^3(s^2+1)} = \frac{s^2+1+s^4-s^4-s^2}{s^3(s^2+1)} = \frac{1}{s^3(s^2+1)} = F(s)$$

$$(15) \mathcal{L}^{-1} \left\{ \frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{3}{s-\frac{3}{2}} - \frac{3}{9(s^2-\frac{16}{9})} - \frac{4}{9} \frac{s}{(s^2-\frac{16}{9})} + \frac{8}{16(s^2+\frac{9}{16})} - \frac{6}{16} \frac{s}{(s^2+\frac{9}{16})} \right\}$$

$$= \mathcal{L}^{-1} \frac{3}{s-\frac{3}{2}} - \frac{1}{3} \mathcal{L}^{-1} \frac{1}{s^2-\frac{16}{9}} - \frac{4}{9} \mathcal{L}^{-1} \frac{s}{s^2-\frac{16}{9}} + \frac{1}{2} \mathcal{L}^{-1} \frac{1}{s^2+\frac{9}{16}} - \frac{3}{8} \mathcal{L}^{-1} \frac{s}{s^2+\frac{9}{16}}$$

$$3e^{\frac{3}{2}t} - \frac{1}{4} \sinh \frac{4}{3}t - \frac{4}{9} \cosh \frac{4}{3}t + \frac{2}{3} \sin \frac{3}{4}t - \frac{3}{8} \cos \frac{3}{4}t$$

$$1) \mathcal{L}^{-1} \frac{1}{2s-5} = \mathcal{L}^{-1} \frac{1}{2(s-\frac{5}{2})} = 2 \mathcal{L}^{-1} \frac{1}{s-\frac{5}{2}} = 2e^{+\frac{5}{2}t}$$

$$2) \mathcal{L}^{-1} \frac{12}{4-3s} = \mathcal{L}^{-1} \frac{12}{-3s+4} = \mathcal{L}^{-1} \frac{12}{-3(s-\frac{4}{3})} = \mathcal{L}^{-1} \frac{-4}{s-\frac{4}{3}} \\ -4 \mathcal{L}^{-1} \frac{1}{s-\frac{4}{3}} = -4e^{+\frac{4}{3}t}$$

$$3) \mathcal{L}^{-1} \frac{3(s^2-1)^2}{2s^5} = \frac{3}{2} \mathcal{L}^{-1} \frac{(s^2-1)^2}{s^5} = \frac{3}{2} \mathcal{L}^{-1} \frac{s^4-2s^2+1}{s^5} \\ \frac{3}{2} \mathcal{L}^{-1} \left(\frac{s^4}{s^5} - \frac{2s^2}{s^5} + \frac{1}{s^5} \right) = \frac{3}{2} \mathcal{L}^{-1} \left(\frac{1}{s} - \frac{2}{s^3} + \frac{1}{s^5} \right) \\ \frac{3}{2} \left(1 - t^2 + \frac{1}{24} t^4 \right)$$

$$4) \mathcal{L}^{-1} \frac{4s-18}{9-s^2} = \mathcal{L}^{-1} \frac{4s-18}{-s^2+9} = -\mathcal{L}^{-1} \left(\frac{4s-18}{s^2-9} \right) \\ = -\mathcal{L}^{-1} \left(\frac{4s}{s^2-9} - \frac{18}{s^2-9} \right) = -4 \mathcal{L}^{-1} \frac{s}{s^2-9} + \mathcal{L}^{-1} \frac{6 \times 3}{s^2-9} \\ -4 \cosh 3t + 6 \sinh 3t$$