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Ex Solve  $y'' + 2y = \cos t$ ,  $y=1$ , at  $t=0$   $\{y(0)=1\}$  "(s)"

Solution take laplace trans. for both sides

$$sY(s) - y(0) + 2Y(s) = \frac{s}{s^2+1}$$

$$sY(s) - 1 + 2Y(s) = \frac{s}{s^2+1} \Rightarrow Y(s)(s+2) - 1 = \frac{s}{s^2+1}$$

$$Y(s)(s+2) = \frac{s}{s^2+1} + 1 \Rightarrow Y(s) = \frac{s^2+s+1}{(s^2+1)(s+2)}$$

$$= \frac{As+B}{s^2+1} + \frac{C}{s+2} = \frac{(As+B)(s+2) + C(s^2+1)}{(s^2+1)(s+2)}$$

$$s^2+s+1 = As^2 + 2As + Bs + 2B + Cs^2 + C$$

$$B = \frac{1}{5}, A = \frac{2}{5}, C = \frac{3}{5}$$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{\frac{2}{5}s + \frac{1}{5}}{s^2+1} + \frac{\frac{3}{5}}{s+2} \right\}$$

$$A + C = 1$$

$$2A + B = 1$$

$$2B + C = 1$$

$$y(t) = \frac{2}{5} \mathcal{L}^{-1} \frac{s}{s^2+1} - \frac{1}{5} \mathcal{L}^{-1} \frac{1}{s^2+1} + \frac{3}{5} \mathcal{L}^{-1} \frac{1}{s+2}$$

$$= \frac{2}{5} \cos t - \frac{1}{5} \sin t + \frac{3}{5} e^{-2t}$$

OR we can solve this example using complex roots  $\Rightarrow$

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$$1(s) = \frac{s^2 + s + 1}{(s^2 + 1)(s + 2)} = \frac{s^2 + s + 1}{(s - i)(s + i)(s + 2)} = \frac{A}{s - i} + \frac{B}{s + i} + \frac{C}{s + 2}$$

$$A = \frac{s^2 + s + 1}{(s - i)(s + i)(s + 2)} \bigg|_{s=i} = \frac{i}{2i(-i+2)} = \frac{1}{2i+4}$$

$$B = \frac{s^2 + s + 1}{(s - i)(s + i)(s + 2)} \bigg|_{s=-i} = \frac{-i}{-2i(-i+2)} = \frac{1}{-2i+4}$$

$$C = \frac{s^2 + s + 1}{(s - i)(s + i)(s + 2)} \bigg|_{s=-2} = \frac{3}{5}$$

$$\therefore y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{2i+4} \frac{1}{s-i} + \frac{1}{-2i+4} \frac{1}{s+i} + \frac{3}{5} \frac{1}{s+2} \right\}$$

$$y(t) = \frac{1}{2i+4} e^{it} + \frac{1}{-2i+4} e^{-it} + \frac{3}{5} e^{-2t}$$

Ex Solve the following

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$$y'''' + 2y''' + 2y'' + 2y' + y = e^{-t},$$

$$y(0) = y'(0) = y''(0) = y'''(0) = 0$$

Solu Take  $\mathcal{L}$  on both sides and substitute the initial conditions

$$s^4 \mathcal{L}y(t) + 2s^3 \mathcal{L}y(t) + 2s^2 \mathcal{L}y(t) + 2s \mathcal{L}y(t) + \mathcal{L}y(t) = \frac{1}{s+1}$$

$$Y(s)(s^4 + 2s^3 + 2s^2 + 2s + 1) = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)(s^4 + 2s^3 + 2s^2 + 2s + 1)}$$

$$Y(s) = \frac{1}{(s+1)(s+1)^2(s^2+1)} = \frac{1}{(s+1)^3(s^2+1)}$$

take  $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{A}{(s+1)^3} + \frac{B}{(s+1)^2} + \frac{C}{s+1} + \frac{Ds+E}{s^2+1} \right\}$$

$$A = \frac{1}{2}, B = \frac{1}{2}, C = \frac{1}{4}, D = -\frac{1}{4}, E = -\frac{1}{4}$$

$$\begin{aligned} y(t) &= \frac{1}{2} \mathcal{L}^{-1} \frac{1}{(s+1)^3} + \frac{1}{2} \mathcal{L}^{-1} \frac{1}{(s+1)^2} + \frac{1}{4} \mathcal{L}^{-1} \frac{1}{s+1} - \frac{1}{4} \mathcal{L}^{-1} \frac{s}{s^2+1} - \frac{1}{4} \mathcal{L}^{-1} \frac{1}{s^2+1} \\ &= \frac{1}{4} t^2 e^{-t} + \frac{1}{2} t e^{-t} + \frac{1}{4} e^{-t} - \frac{1}{4} (\cos t + \sin t) \end{aligned}$$

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examples

1.  $y'' - 2y' + 2y = e^{-t}$        $y(0) = 0, y'(0) = 1$

2.  $y'' + y = \cos 2t$        $y(0) = 2, y'(0) = 1$

3.  $f''(t) - f(t) = t$       where  $f(0) = 1, f'(0) = -2$

4.  $y'' + 5y' + 6y = 3$        $y(0) = 2, y'(0) = 0$

5.  $y'' + 2y' + y = \sin t$        $y(0) = 3, y'(0) = 1$

6.  $f''(t) - f'(t) - 2f(t) = e^{-t} \sin 2t$        $f(0) = 0, f'(0) = 2$

7.  $y'' + 4y' + 5y = 10e^t$        $y(0) = 1, y'(0) = 2$

Ex Solve  $y'''' + 2y''' + 2y'' + 2y' + y = e^{-t}$

Solution: with  $y(0) = y'(0) = y''(0) = y'''(0) = 0$

$$\mathcal{L} y'''' + 2\mathcal{L} y''' + 2\mathcal{L} y'' + 2\mathcal{L} y' + \mathcal{L} y = \mathcal{L} e^{-t}$$

$$S^4 Y(s) - S^3 y(0) - S^2 y'(0) - S y''(0) - y'''(0)$$

$$+ 2(S^3 Y(s) - S^2 y(0) - S y'(0) - y''(0))$$

$$+ 2(S^2 Y(s) - S y(0) - y'(0))$$

$$+ 2(S Y(s) - y(0)) + Y(s) = \frac{1}{S+1}$$

$$= S^4 Y(s) + 2S^3 Y(s) + 2S^2 Y(s) + 2S Y(s) + Y(s) = \frac{1}{S+1}$$

$$Y(s) (S^4 + 2S^3 + 2S^2 + 2S + 1) = \frac{1}{S+1}$$

$$\therefore Y(s) = \frac{1}{(S+1)(S^4 + 2S^3 + 2S^2 + 2S + 1)}$$

$$\text{if } S=0 \Rightarrow (S^4 + 2S^3 + 2S^2 + 2S + 1) = 1 \quad (\text{done})$$

$$\text{if } S=1 \Rightarrow (S^4 + 2S^3 + 2S^2 + 2S + 1) = 8 \quad (\text{done})$$

$$\text{if } S=-1 \Rightarrow (S^4 + 2S^3 + 2S^2 + 2S + 1) = 0$$

$$\therefore S=-1 \text{ (is solution)} \Rightarrow (S+1)$$

$$\begin{aligned} & \cdot \sqrt{(S^4 + 2S^3 + 2S^2 + 2S + 1)} \\ & = (S+1)(S^3 + S^2 + S + 1) \end{aligned}$$

$$\Rightarrow \frac{1}{(S+1)(S+1)(S^3 + S^2 + S + 1)}$$

i f  $S = -1 \Rightarrow (S^3 + S^2 + S + 1) = 0 \Rightarrow (S+1)$  is a solution

$$\Rightarrow (S^3 + S^2 + S + 1) = (S+1)(S^2 + 1)$$

$$\Rightarrow \frac{1}{(S+1)(S+1)(S+1)(S^2+1)} = \frac{1}{(S+1)^3(S^2+1)}$$

$$\Rightarrow \mathcal{L}^{-1} \frac{1}{(S+1)^3(S^2+1)} = \frac{A}{(S+1)^3} + \frac{B}{(S+1)^2} + \frac{C}{(S+1)} + \frac{DS+E}{(S^2+1)}$$

$$A = \frac{1}{2}, B = \frac{1}{2}, C = \frac{1}{4}, D = -\frac{1}{4}, E = -\frac{1}{4}$$

$$\Rightarrow y(t) = \frac{1}{2} \mathcal{L}^{-1} \frac{1}{(S+1)^3} + \frac{1}{2} \mathcal{L}^{-1} \frac{1}{(S+1)^2} + \frac{1}{4} \mathcal{L}^{-1} \frac{1}{S+1} - \frac{1}{4} \mathcal{L}^{-1} \frac{S}{S^2+1} - \frac{1}{4} \mathcal{L}^{-1} \frac{1}{S^2+1}$$

$$y(t) = \frac{1}{2} e^{-t} t^2 + \frac{1}{2} e^{-t} t + \frac{1}{4} e^{-t} - \frac{1}{4} \cos t - \frac{1}{4} \sin t,$$

$$\begin{array}{r} S^3 + S^2 + S + 1 \\ S+1 \overline{) S^4 + 2S^3 + 2S^2 + 2S + 1} \\ \underline{S^4 + S^3} \phantom{+ 2S^2 + 2S + 1} \\ S^3 - 2S^2 + 2S + 1 \\ \underline{S^3 + S^2} \phantom{+ 2S + 1} \\ S^2 + 2S + 1 \\ \underline{S^2 + S} \phantom{+ 1} \\ S + 1 \end{array}$$

$$\begin{array}{r} S^2 + 1 \\ S+1 \overline{) S^3 + S^2 + S + 1} \\ \underline{S^3 + S^2} \phantom{+ S + 1} \\ S + 1 \\ \underline{S + 1} \\ 0 \end{array}$$