

Partial Fraction method طريقة الكسور الجزئية

$$\frac{f(s)}{g(s)}$$

$$g(s) \neq 0$$

1) $g(s) = (as + b)(cs + d)(es + f) \dots$

$$\frac{A}{(as + b)} + \frac{B}{(cs + d)} + \frac{C}{(es + f)} + \dots$$

Ex $\frac{\quad}{(s-2)(s+4)(s-1)} = \frac{A}{s-2} + \frac{B}{s+4} + \frac{C}{s-1}$

2) $g(s) = s^n (as + b)$

$$\frac{A}{s^n} + \frac{B}{s^{n-1}} + \frac{C}{s^{n-2}} + \dots + \frac{k}{as + b}$$

Ex $\frac{\quad}{s^3(s+5)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+5}$

3) $g(s) = (as^2 + bs + d)(es + f)$

$$\frac{As + B}{as^2 + bs + d} + \frac{C}{es + f}$$

Ex $\frac{\quad}{(3s^2 - s + 4)(s + 3)} = \frac{As + B}{3s^2 - s + 4} + \frac{C}{s + 3}$

Partial Fraction method

Ex 16 Find $\mathcal{L}^{-1} \left\{ \frac{1}{s^2 - 5s + 6} \right\}$

Solution:

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2 - 5s + 6} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{(s-3)(s-2)} \right\} =$$

$$\mathcal{L}^{-1} \left(\frac{A}{s-3} + \frac{B}{s-2} \right) = \mathcal{L}^{-1} \frac{A(s-2) + B(s-3)}{(s-3)(s-2)}$$

$$= \mathcal{L}^{-1} \frac{As - 2A + Bs - 3B}{(s-3)(s-2)} = \mathcal{L}^{-1} \frac{s(A+B) + (-2A - 3B)}{(s-3)(s-2)}$$

$$\frac{1}{(s-3)(s-2)} = \frac{s(A+B) + (-2A - 3B)}{(s-3)(s-2)}$$

$$-2A - 3B = 1 \quad \text{--- (1)}$$

$$A + B = 0 \Rightarrow \boxed{A = -B} \quad \text{تعويض معادلة (1)}$$

$$-2(-B) - 3B = 1 \Rightarrow 2B - 3B = 1 \Rightarrow \boxed{B = -1}, A = 1$$

$$\Rightarrow \mathcal{L}^{-1} \frac{1}{s-3} + \frac{-1}{s-2} = e^{3t} - e^{2t}$$

Ex. Find $\mathcal{L}^{-1} \left\{ \frac{3s+7}{s^2-2s-3} \right\}$

Solution

$$\mathcal{L}^{-1} \left\{ \frac{3s+7}{s^2-2s-3} \right\} = \mathcal{L}^{-1} \left\{ \frac{3s+7}{(s-3)(s+1)} \right\} = \mathcal{L}^{-1} \left(\frac{A}{s-3} + \frac{B}{s+1} \right)$$

$$= \mathcal{L}^{-1} \frac{A(s+1) + B(s-3)}{(s-3)(s+1)} = \mathcal{L}^{-1} \frac{As + A + Bs - 3B}{(s-3)(s+1)}$$

$$= \mathcal{L}^{-1} \frac{s(A+B) + (A-3B)}{(s-3)(s+1)} \Rightarrow 3s+7 = s(A+B) + (A-3B)$$

$$\begin{aligned} A+B &= 3 \\ A-3B &= 7 \end{aligned} \quad \text{eqn.} \Rightarrow \begin{aligned} A+B &= 3 \\ -A+3B &= -7 \end{aligned} \Rightarrow 4B = -4 \Rightarrow \boxed{B = -1}$$

$$= \mathcal{L}^{-1} \frac{4}{s-3} + \frac{-1}{s+1} = 4e^{3t} - e^{-t}$$

$$\boxed{A=4}$$

or

$$\mathcal{L}^{-1} \frac{3s+7}{s^2-2s-3} = \mathcal{L}^{-1} \frac{3s+7}{(s-3)(s+1)} = \mathcal{L}^{-1} \frac{A}{s-3} + \frac{B}{s+1}$$

$$A = \frac{3s+7}{(s-3)(s+1)} \Big|_{s=3} = \frac{16}{4} = \boxed{4 = A}$$

$$B = \frac{3s+7}{(s-3)(s+1)} \Big|_{s=-1} = \frac{4}{-4} = \boxed{-1 = B}$$

$$\therefore \mathcal{L}^{-1} \frac{4}{s-3} + \frac{-1}{s+1} = 4e^{3t} - e^{-t}$$

Ex Solve $\mathcal{L}^{-1} \frac{2s^2 - 16}{s^3 - 16s}$ Using Partial Fraction

Sol: $\mathcal{L}^{-1} \frac{2s^2 - 16}{s(s^2 - 16)} = \mathcal{L}^{-1} \frac{2s^2 - 16}{s(s-4)(s+4)} = \mathcal{L}^{-1} \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s+4}$

$$= \mathcal{L}^{-1} \frac{A(s^2 - 16) + Bs(s+4) + Cs(s-4)}{s(s-4)(s+4)}$$

$$= \mathcal{L}^{-1} \frac{As^2 - 16A + Bs^2 + 4Bs + Cs^2 - 4Cs}{s(s-4)(s+4)}$$

$$A + B + C = 2 \quad \text{--- (1)}$$

$$4B - 4C = 0 \quad \text{--- (2)}$$

$$-16A = -16 \quad \text{--- (3)} \Rightarrow \boxed{A=1} \text{ Put in (1)}$$

$$B + C = 1 \Rightarrow$$

$$4B - 4C = 0 \div 4 \Rightarrow B - C = 0 \quad + \quad 2B = 1 \Rightarrow \boxed{B = \frac{1}{2}}, \boxed{C = \frac{1}{2}}$$

$$\therefore \mathcal{L}^{-1} \frac{1}{s} + \frac{\frac{1}{2}}{s-4} + \frac{\frac{1}{2}}{s+4} = 1 + \frac{1}{2} e^{4t} + \frac{1}{2} e^{-4t}$$

$$\underline{\text{or}} \quad \mathcal{L}^{-1} \frac{2s^2 - 16}{s(s-4)(s+4)} = \mathcal{L}^{-1} \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s+4}$$

$$A = \frac{2s^2 - 16}{s(s-4)(s+4)} \bigg|_{s=0} = \frac{-16}{-16} = \boxed{1 = A}$$

$$B = \frac{2s^2 - 16}{s(s-4)(s+4)} (s-4) \bigg|_{s=4} = \frac{16}{32} = \boxed{\frac{1}{2} = B}$$

$$C = \frac{2s^2 - 16}{s(s-4)(s+4)} (s+4) \bigg|_{s=-4} = \frac{16}{32} = \boxed{\frac{1}{2} = C}$$

$$\therefore \mathcal{L}^{-1} \frac{1}{s} + \frac{\frac{1}{2}}{s-4} + \frac{\frac{1}{2}}{s+4} = 1 + \frac{1}{2} e^{4t} + \frac{1}{2} e^{-4t}$$