

Examples

① Find $F(s)$ if $f(t) = t e^{-2t}$

Solution

Using $\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s)$

① $F(s) \Leftarrow \mathcal{L} f(t)$ is given

② $\left[\frac{d}{ds} = -t \right] F(s)$ is given

③ $(-1)^n$ is given

① $\mathcal{L} e^{-2t} = \frac{1}{s+2} = F(s)$

$$F'(s) = \frac{(s+2)(0) - 1(1)}{(s+2)^2} = \frac{-1}{(s+2)^2}$$

③ (-1) is given

$$\Rightarrow \mathcal{L} t e^{-2t} = \frac{-1}{(s+2)^2} \times -1 = \frac{1}{(s+2)^2}$$

② Find $\mathcal{L} t \sin(2t)$

$$① \mathcal{L} \sin 2t = \frac{2}{s^2+4} = F(s)$$

$$② F'(s) = \frac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2} = \frac{-4s}{(s^2+4)^2}$$

③ (-1) is given

$$\Rightarrow \mathcal{L} t \sin(2t) = \frac{4s}{(s^2+4)^2}$$

3) If $\mathcal{L}[f(t)] = \frac{s}{s^3+2}$ Find $\mathcal{L}[e^{-t}f(2t)]$

Sol:

$\mathcal{L}[e^{at}f(t)] = F(s-a) \Rightarrow$ (1) $\mathcal{L}[f(t)] = F(s)$
 (2) $(s-a) \rightarrow s$ is replaced by

$\therefore \mathcal{L}[f(2t)] = ?$

$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right) \Rightarrow \mathcal{L}[f(2t)] = \frac{1}{2} F\left(\frac{s}{2}\right)$

$\therefore F(s) = \frac{s}{s^3+2} \Rightarrow \mathcal{L}[f(2t)] = \frac{1}{2} \frac{\frac{s}{2}}{\left(\frac{s}{2}\right)^3+2}$

$= \frac{1}{4} \frac{s}{\frac{s^3}{8}+2} = \frac{1}{4} \frac{s}{\frac{s^3+16}{8}} = \frac{1}{4} \frac{8s}{s^3+16} = \frac{2s}{s^3+16}$

$\therefore \mathcal{L}[e^{-t}f(2t)] = \frac{2(s+1)}{(s+1)^3+16}$

4) $\mathcal{L}[t^4 - 4t + 5 + 3\sin 2t]$

Sol: $\mathcal{L}[t^4 - 4t + 5 + 3\sin 2t]$
 $= \frac{4!}{s^5} - 4 \frac{1}{s^2} + \frac{5}{s} + 3 \frac{2}{s^2+4} =$

$\frac{24}{s^5} - \frac{4}{s^2} + \frac{5}{s} + \frac{6}{s^2+4}$

5) $\mathcal{L} \{ t \cos 7t \}$

Solu $\mathcal{L} \{ t \cos 7t \} =$

① $\mathcal{L}(\cos 7t) = \frac{s}{s^2 + 49}$ ② $F'(s) = \frac{(s^2 + 49)(1) - s(2s)}{(s^2 + 49)^2}$

$F'(s) = \frac{s^2 + 49 - 2s^2}{(s^2 + 49)^2} = \frac{-s^2 + 49}{(s^2 + 49)^2} \times (-1)$

$= \frac{s^2 - 49}{(s^2 + 49)^2} \Rightarrow \mathcal{L} \{ t \cos 7t \} = \frac{s^2 - 49}{(s^2 + 49)^2}$

6) Find $\mathcal{L} \cos^2 t$

Solu

$\cos^2 t = \frac{1 + \cos 2t}{2} = \frac{1}{2} (1 + \cos 2t)$

$\therefore \mathcal{L} \frac{1}{2} (1 + \cos 2t) = \frac{1}{2} [\mathcal{L}(1) + \mathcal{L} \cos(2t)]$

$= \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2 + 4} \right] = \frac{1}{2} \frac{s^2 + 4 + s^2}{s(s^2 + 4)} = \frac{1}{2} \frac{2s^2 + 4}{s(s^2 + 4)} = \frac{2}{2} \frac{s^2 + 2}{s(s^2 + 4)}$

$= \frac{s^2 + 2}{s(s^2 + 4)}$

8) $\mathcal{L} \int_0^t \sin 2u \, du$?

Solu $\mathcal{L} \int_0^t f(u) \, du = \frac{F(s)}{s}$ (1) $\mathcal{L} f(t) = F(s)$

$\therefore$ (1) $\mathcal{L} \sin 2t = \frac{2}{s^2 + 4} = F(s)$ (2) $\frac{1}{s}$ is use

(2) $\frac{F(s)}{s} = \frac{2}{s^2 + 4} \times \frac{1}{s} = \frac{2}{s(s^2 + 4)}$

9) $\mathcal{L} \{t^2 e^{2t}\}$

Solu (1) $\mathcal{L} e^{2t} = \frac{1}{s-2}$

(2) $F'(s) = \frac{-1}{(s-2)^2} \Rightarrow F''(s) = \frac{(s-2)^2(0) - (-1)[2(s-2)]}{(s-2)^4}$

$F''(s) = \frac{2(s-2)}{(s-2)^4} = \frac{2}{(s-2)^3}$ (3) $1 = (-1)^2$ is use

$\therefore \mathcal{L} t^2 e^{2t} = \frac{2}{(s-2)^3}$

10) Find ($\mathcal{L}$) transform of the following functions:

(a) $f(t) = \int_0^t \cos(4t) \, dt$

Solu $f(t) = \cos(4t) \Rightarrow \mathcal{L} f(t) = \frac{s}{s^2 + 16} = F(s)$

$\therefore \frac{F(s)}{s} \Rightarrow \frac{s}{s^2 + 16} \times \frac{1}{s} = \frac{1}{s^2 + 16}$

$$b) f(t) = \int_0^t e^{2t} \cos(3t) dt$$

Soln

$$\mathcal{L} \int_0^t f(t) dt = \frac{F(s)}{s}$$

$$① \mathcal{L} f(t) \Rightarrow F(s)$$

$$\therefore f(t) = e^{2t} \cos(3t) \Rightarrow \mathcal{L} f(t) = \mathcal{L} e^{2t} \cos(3t)$$

$$\mathcal{L} e^{\frac{s}{s^2+9}} = \frac{s-2}{(s-2)^2+9} = F(s)$$

$$\therefore \frac{F(s)}{s} = \frac{(s-2)}{s[(s-2)^2+9]}$$

$$c) f(t) = \int_0^t t e^{-3t} dt$$

Sol

$$f(t) = t e^{-3t} \Rightarrow \mathcal{L} f(t) = \mathcal{L} t e^{-3t} = \mathcal{L} t \frac{1}{s+3}$$

$$\Rightarrow \frac{-1}{(s+3)^2} \Rightarrow (-1) \neq 2$$

$$\Rightarrow \mathcal{L} t e^{-3t} = \frac{1}{(s+3)^2} = F(s)$$

$$\therefore \frac{F(s)}{s} = \frac{1}{s(s+3)^2}$$

$$d) f(t) = \int_0^t \sin(at) \cos(at) dt$$

Solu

$$\text{We have } \sin 2t = 2 \sin t \cos t$$

$$\rightarrow \sin(2at) = 2 \sin(at) \cos(at) \quad \div 2$$

$$\frac{\sin(2at)}{2} = \sin(at) \cos(at)$$

$$\rightarrow f(t) = \int_0^t \frac{\sin 2at}{2} dt$$

$$\rightarrow \mathcal{L} \frac{\sin(2at)}{2} = \frac{1}{2} \mathcal{L} \sin(2at) = \frac{1}{2} \frac{2a}{s^2 + 4a^2}$$

$$= \frac{a}{s^2 + 4a^2} = F(s)$$

$$\therefore \frac{F(s)}{s} = \frac{a}{s(s^2 + 4a^2)}$$

H. w

$$\mathcal{L} \{ e^{4t} \cosh(5t) \}, \mathcal{L} \{ e^{-2t} (3 \cos 6t - 5 \sin 6t) \}$$