

$$\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a}}$$

Ex

$$1) \mathcal{L}(e^{2t}) = \frac{1}{s-2}$$

$$2) \mathcal{L}(e^{-3t}) = \frac{1}{s+3}$$

$$3) \mathcal{L}(e^t) = \frac{1}{s-1}$$

$$4) \mathcal{L}(e^{\frac{-2t}{3}}) = \frac{1}{s+\frac{2}{3}}$$

$$5) \mathcal{L}(9e^{\pi t}) = 9\mathcal{L}(e^{\pi t}) = 9\left(\frac{1}{s-\pi}\right) = \frac{9}{s-\pi}$$

$$6) \mathcal{L}\left(\frac{3}{2}t^3\right) = \frac{3}{2}\mathcal{L}(t^3) = \frac{3}{2} \cdot \frac{3!}{s^4} = \frac{3}{2} \cdot \frac{6}{s^4} = \frac{9}{s^4}$$

$$7) \mathcal{L}\left(3\cos t - \frac{\sinh 4t}{2} + 5e^{-\sqrt{3}t}\right)$$

$$= 3 \frac{s}{s^2+1} - \frac{1}{2} \frac{4}{s^2-16} + 5 \frac{1}{s+\sqrt{3}} = \frac{3s}{s^2+1} - \frac{2}{s^2-16} + \frac{5}{s+\sqrt{3}}$$

Elementary Properties of Laplace Transform

Suppose $a, b \in \mathbb{C}$ and f, g functions for which Laplace Transform exists then:-

1. $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$ [Linearity]

2. $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ [First shifting property]

3. Suppose f and all its derivatives up to and including order " n " are continuous on $[0, \infty)$ and each derivative has a Laplace transformation, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

4. If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\left\{\int_0^t f(u) du\right\} = \frac{F(s)}{s}$

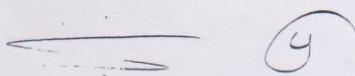
5. $\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$ [Time scaling property]

6. First derivative: $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

Second derivative: $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

and higher order derivatives:-

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$



$$\boxed{\mathcal{L}(e^{\pm at} f(t)) = F(s \mp a)}$$

① $F(s) \leftrightarrow \mathcal{L} f(t)$ نأخذ

② $(s \pm a)$ يُبدل كل s بـ $s \mp a$
 ← عكس الإشارة

Ex

1) $\mathcal{L} e^{-t} \cos \sqrt{2} t =$

① $\mathcal{L} \cos \sqrt{2} t = \frac{s}{s^2 + 2}$

$\therefore \mathcal{L} e^{-t} \cos \sqrt{2} t = \frac{s+1}{(s+1)^2 + 2}$

2) $\mathcal{L} (t^4 e^t)$

$\because \mathcal{L} t = \frac{1}{s^2} \Rightarrow \mathcal{L} t^4 e^t = \frac{1}{(s-1)^2}$

$$\boxed{\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)}$$

① $\mathcal{L} f(t)$ نأخذ

② $F(s)$ نشتق بعدد n مرّات
 في القانون [بالعدد n]

③ $(-1)^n$ بـ $(-1)^n$ نكتب

$\mathcal{L} t^2 \sin 3t =$

① $\mathcal{L} \sin 3t = \frac{3}{s^2 + 9} = F(s)$

② $\Rightarrow F'(s) = \frac{(s^2+9)(0) - 3(2s)}{(s^2+9)^2} = \frac{-6s}{(s^2+9)^2}$
 $F''(s) = \frac{(s^2+9)^2(-6) - (-6s)(2(s^2+9)(2s))}{(s^2+9)^4}$

$$\Rightarrow \bar{F}(s) = \frac{(s^2+9) [(s^2+9)(-6) + 24s^2]}{(s^2+9)^4} = \frac{-6s^2 - 54 + 24s^2}{(s^2+9)^3}$$

$$\bar{F}(s) = \frac{18s^2 - 54}{(s^2+9)^3}$$

3) $\bar{f}(s)$ بـ $\frac{1}{s^2+9}$ انصاف

$$(-1)^2 = 1$$

$$\Rightarrow \mathcal{L} t^2 \sin 3t = \frac{18s^2 - 54}{(s^2+9)^3} \times 1$$

$$\boxed{\mathcal{L} \left\{ \int_0^t f(u) du \right\} = \frac{F(s)}{s}}$$

$$1) \mathcal{L} f(t) = F(s) \text{ نالو}$$

$$2) \text{ تقسم اوله } F(s) \text{ لك } s \text{ او } \bar{f}(s) = \left(\frac{1}{s} \right)$$

$$\mathcal{L} \int_0^t 7u e^u du \Rightarrow \mathcal{L} 7t e^t = \frac{1}{s-7} = F(s)$$

$$F(s) = \frac{1}{s-7} = \frac{1}{s} \leftarrow \frac{1}{s} \Rightarrow F(s) \text{ اوله } \bar{f}(s) = \frac{1}{s(s-7)}$$

$$2) \mathcal{L} \int_0^t \cos 6t dt$$

$$\mathcal{L} \cos 6t = \frac{s}{s^2+36} = F(s) \Rightarrow \frac{F(s)}{s} = \frac{s}{s^2+36} \times \frac{1}{s}$$

$$\Rightarrow \mathcal{L} \int_0^t \cos 6t dt = \frac{1}{s^2+36}$$

$$5) \mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right) \quad (\text{Time Scaling property})$$

① $\mathcal{L} f(t) = F(s)$ ② $\frac{s}{a}$ پہلے s کی جگہ پر ③ $\frac{1}{a}$ پہلے s کی جگہ پر

Ex $\mathcal{L} \cos 3t = \frac{s}{s^2 + 9}$

or $f(at) = f(3t) = \cos 3t$

$f(t) = \cos t \Rightarrow \mathcal{L} f(t) = \mathcal{L} \cos t = \frac{s}{s^2 + 1}$

② $\frac{s}{3}$ پہلے s کی جگہ پر $\Rightarrow \frac{\frac{s}{3}}{(\frac{s}{3})^2 + 1} =$

$\frac{\frac{s}{3}}{\frac{s^2}{9} + 1} = \frac{\frac{s}{3}}{\frac{s^2 + 9}{9}} = \frac{\frac{9s}{3}}{s^2 + 9} = \frac{3s}{s^2 + 9}$

③ $\frac{1}{3}$ پہلے s کی جگہ پر $\Rightarrow \frac{1}{3} + \frac{3s}{s^2 + 9} = \frac{s}{s^2 + 9}$

6) First derivative

$$\mathcal{L} f'(t) = s \mathcal{L} f(t) - f(0) = [sF(s) - f(0)]$$

$$\mathcal{L} f''(t) = s^2 \mathcal{L} f(t) - s f(0) - f'(0) = [s^2 F(s) - s f(0) - f'(0)]$$

$$\mathcal{L} f'''(t) = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$\mathcal{L} f^{(4)}(t) = s^4 F(s) - s^3 f(0) - s^2 f'(0) - s f''(0) - f^{(3)}(0)$$

$$\mathcal{L} f^{(n)}(t) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$