

The Solution of Linear System of Equations Numerically

A general system of m linear equation with n unknown can be written as,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$\vdots$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

here $x_1, x_2, x_3, \dots, x_n$ are the unknowns
 $a_{11}, a_{12}, \dots, a_{mn}$ are the coefficients
 of the system and $b_1, b_2, \dots, b_m$ are
 the constant terms.

Matrix Equation

The matrix equation of the above linear system is

$$Ax = b$$

where A is an $(m \times n)$ matrix, x is a column vector with n entries and b is a column vector with m entries

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$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A system of linear eq. is homogenous if all the constant terms are zero ($b=0$)

$$AX=0$$

Numerical Methods

Numerical methods for solving linear system of eq. can generally be divided into two classes

- * Direct methods: In the absence of round off error such methods would yield the exact solution within a finite number of steps such as LU method and Gaussian elimination method, and Gauss-Jordan method.
- * Iterative methods: There are methods that are useful for problems involving special, very large matrices, such as, Jacobi and Gauss-Seidel iterations method.

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1. Gaussian Elimination Method

The goal of this method is to convert the coefficients matrix "A" into an upper triangular matrix using forward elimination and then using back substitution to find " x_i ", it is easy to illustrate this method by solving an example.

Ex Solve

$$\begin{aligned} 2x_2 + x_4 &= 0 \\ 2x_1 + 2x_2 + 3x_3 + 2x_4 &= -2 \\ 4x_1 - 3x_2 + x_4 &= -7 \\ 6x_1 + x_2 - 6x_3 - 5x_4 &= 6 \end{aligned}$$

using Gauss elimination [four decimal places]

Solution:

* Form the matrix (A|b)

$$(A|b) = \left(\begin{array}{cccc|c} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{array} \right)$$

* We note that $a_{11} = 0$ and using Gauss-elimination we will have division by zero error B

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to avoid this error we will use the "Partial pivot" ~~method~~, by rearrange the equations (rows), we put " a_{11} " the largest element regardless the sign

$$\begin{pmatrix} 6 & 1 & -6 & -5 & 6 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{pmatrix}$$

* put $a_{i1} = 0 \quad i = 2, 3, 4$

using $R_i = R_i - (a_{i1}/a_{11})R_1$

$$\begin{pmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & 1.6666 & 5 & 3.6666 & -4 \\ 0 & -3.6666 & 4 & 4.3333 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{pmatrix}$$

* Replacing R_3 in stead of R_2

$$\begin{pmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6666 & 4 & 4.3333 & -11 \\ 0 & 1.6666 & 5 & 3.6666 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{pmatrix}$$

* put $a_{i2} = 0 \quad i = 3, 4$, $R_i = R_i - (a_{i2}/a_{22})R_2$

$$\begin{pmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6666 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8181 & 5.6362 & -8.9998 \\ 0 & 0 & 2.1818 & 3.3636 & -5.0000 \end{pmatrix}$$

by

$$\left(\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.666 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8181 & 5.6362 & -8.9998 \\ 0 & 0 & 0 & 1.56 & -2.12 \\ & & & & -3.1201 \end{array} \right)$$

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This is an upper triangular matrix
now using back substitution to find x_4

$$x_n = a_{n,n}/a_{nn} \Rightarrow x_4 = \frac{a_{45}}{a_{44}} \Rightarrow (1.56 x_4 = -2.12)$$

$$x_4 = -1.3589$$

$$x_i = (a_{i,n+1} - \sum_{j=i+1}^n a_{ij} x_j) / a_{ii}$$

to find $x_{n-1}, x_{n-2}, \dots, x_1$

$$x_3 = (a_{35} - a_{34} x_4) / a_{33} = \frac{-1.3407}{6.8181} = -0.1966$$

$$x_2 = [a_{25} - (a_{23} x_3 + a_{24} x_4)] / a_{22} = 1.1795$$

$$x_1 = [a_{15} - (a_{12} x_2 + a_{13} x_3 + a_{14} x_4)] / a_{11} = -0.5256$$

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Ex Solve for the value of x_1, x_2 and x_3 using Gaussian-elimination

$$-3x_1 + 2x_2 - x_3 = -1$$

$$6x_1 - 6x_2 + 7x_3 = -7$$

$$3x_1 - 4x_2 + 4x_3 = -6$$

(Four decimal places)

Solution

* Form the matrix (A|b)

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{pmatrix} \Rightarrow \begin{aligned} R_2 &= R_2 - (-\frac{6}{-3})R_1 \\ R_3 &= R_3 - (-1)R_1 \end{aligned}$$

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & -2 & 3 & -7 \end{pmatrix} \Rightarrow R_3 = R_3 - R_2$$

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

$$x_3 = -1$$

$$x_2 = 2$$

$$x_1 = 2$$

$$\left. \begin{aligned} 2x - 6y - 2z &= 5 \\ 8x + 3y - z &= 2 \\ x + y + 4z &= 4 \end{aligned} \right\}$$

H.W

Ex.1 Solve for the value of x_1, x_2 and x_3 using Gaussian-elimination method

$$-3x_1 + 2x_2 - x_3 = -1$$

$$6x_1 - 6x_2 + 7x_3 = -7 \quad (\text{four decimal places})$$

$$3x_1 - 4x_2 + 4x_3 = -6$$

Solution

Form the matrix $(A|b)$

$$\left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{array} \right)$$

* القدر تحويل العناصر أسفل القطر الرئيسي للصفر

(A) أي "صف" أي تكتب المعادلات

Put $a_{i1} = 0 \quad i = 2, 3, 4, \dots \quad [i = 2, 3]$

using $R_i = R_i - (a_{i1}/a_{11})R_1$

now! $i = 2 \Rightarrow a_{21} = 0$ using $R_2 = R_2 - (a_{21}/a_{11})R_1$

$$(a_{21}/a_{11}) = \frac{6}{-3} = \boxed{-2} \Rightarrow R_2 - (-2)R_1 \Rightarrow R_2 = R_2 + 2R_1$$

$$6 + 2(-3) = 0$$

$$-6 + 2(2) = -2$$

$$7 + 2(-1) = 5$$

$$-7 + 2(-1) = -9$$

$$\begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & 3 & -7 \end{pmatrix}$$

now when $i=3$ ($a_{31}=0$)

$$R_3 = R_3 - (a_{31}/a_{11}) R_1$$

$$= R_3 - (3/-3) R_1 = R_3 + R_1$$

$$3 + (-3) = 0$$

$$-4 + 2 = -2$$

$$4 + (-1) = 3$$

$$-6 + (-1) = -7$$

$$\Rightarrow \begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & -2 & 3 & -7 \end{pmatrix}$$

* Put $a_{i2}=0$ $i=3, 4, \dots$ [$i=3$]

$$\text{using } R_i = R_i - (a_{i2}/a_{22}) R_2$$

$$\text{when } i=3 \Rightarrow R_3 = R_3 - (a_{32}/a_{22}) R_2$$

$$= R_3 - (-2/-2) R_2$$

$$R_3 = R_3 - R_2$$

$$0 - 0 = 0$$

$$-2 - (-2) = 0$$

$$3 - 5 = -2$$

$$-7 - (-9) = 2$$

$$\Rightarrow \begin{pmatrix} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -9 \\ 2 \end{pmatrix} \Rightarrow \begin{aligned} -3x_1 + 2x_2 - x_3 &= -1 \\ 0x_1 - 2x_2 + 5x_3 &= -9 \\ 0x_1 + 0x_2 - 2x_3 &= 2 \end{aligned}$$

$$x_1 = 2$$

$$x_2 = 2$$

$$\Rightarrow x_3 = \frac{2}{-2} = -1$$