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Solution Of The Differential Equations Using Laplace Transform

The laplace transform is a method of solving ordinary differential equations (ODE) and initial value problem.

Before we solve any example we must first consider the laplace transform of derivatives.

Laplace Transform Of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) = s^2F(s) - sf(0) - f'(0)$$

and the higher order derivative [the laplace trans. of the derivative $f^{(n)}(t)$] satisfies;

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - s^{n-2}f^{(n-2)}(0) - f^{(n-1)}(0)$$

Differential Equations, Initial Value Problems

We shall now discuss how the laplace transform method solves ODEs and initial value problems, the procedure is best illustrated with an example:-

Example: Solve the (I.V.P.) by laplace Trans.
 $y'' - 3y' - 10y = 2$ with $y(0) = 1, y'(0) = 2$

* Structure of Solution *

* Take laplace transform on both sides, You will get an algebraic equation for Y

Solve the equation to get $Y(s)$

Take inverse transform to get $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

Solution:

1- take laplace transform on both sides

let $\mathcal{L}\{y(t)\} = Y(s)$

$$\mathcal{L}\{y''(t)\} - 3\mathcal{L}\{y'(t)\} - 10\mathcal{L}\{y(t)\} = \mathcal{L}\{2\}$$

$$s^2 Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] - 10Y(s) = \frac{2}{s}$$

$$s^2 Y(s) - s - 2 - 3sY(s) + 3 - 10Y(s) = \frac{2}{s}$$

Now we get an algebraic equation for $Y(s)$

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2- Solve it For $Y(s)$,

$$Y(s)(s^2 - 3s - 10) = \frac{2}{s} + s - 1 \Rightarrow Y(s) = \frac{2 + s^2 - s}{s(s^2 - 3s - 10)}$$

$$\therefore Y(s) = \frac{s^2 - s + 2}{s(s-5)(s+2)}$$

3- Take inverse laplace to get $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

The main technique here is partial Fraction

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s^2 - s + 2}{s(s-5)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s-5} + \frac{C}{s+2}\right\}$$

$$= \frac{A(s^2 - 3s - 10) + Bs(s+2) + Cs(s-5)}{s(s-5)(s+2)}$$

$$s^2 - s + 2 = As^2 - 3As - 10A + Bs^2 + 2Bs + Cs^2 - 5Cs$$

$$A + B + C = 1$$

$$-3A + 2B - 5C = -1$$

$$-10A = 2 \Rightarrow$$

$$\boxed{A = -\frac{1}{5}} \quad \boxed{B = \frac{22}{35}} \quad \boxed{C = \frac{4}{7}}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{-\frac{1}{5}}{s} + \frac{\frac{22}{35}}{s-5} + \frac{\frac{4}{7}}{s+2}\right\}$$

$$\therefore y(t) = -\frac{1}{5} + \frac{22}{35} e^{5t} + \frac{4}{7} e^{-2t}$$