

Chapter three

Lec 6

Frequency-Domain Analysis

3.1 Introduction :-

In previous chapter we pointed out that any arbitrary sequence can be represented in the time-domain as a weighted linear combination of delayed unit sample sequences ($\delta[n-k]$). We consider in this chapter an alternate description of a sequence in terms of complex exponential sequences of the form $(e^{-j\omega n})$ and (z^{-n}) . This leads to two particularly useful representations of discrete sequences and systems in a transform domain :-

- 1- Discrete Fourier transform (DFT).
- 2- Z-transform.

3.2 The Discrete Fourier Transform :-

In many signal processing applications, the distinguishing features of signals and systems are most easily interpreted in the frequency domain. The main analytic tool used to transform from time domain ($x[n]$) to frequency domain ($X[k]$) is the Fourier transform. The DFT is important for two reasons: First, it allows us to determine the frequency content of a signal, that is, to perform spectral analysis. The second is to perform filter operations in

the frequency domain. Periodic sequences can be represented in the frequency domain by means of a "discrete Fourier series", which is given by :-

$$\tilde{X}[K] = \sum_{n=0}^{N-1} \tilde{X}[n] e^{-j2\pi Kn/N} \quad \dots\dots\dots (3.1)$$

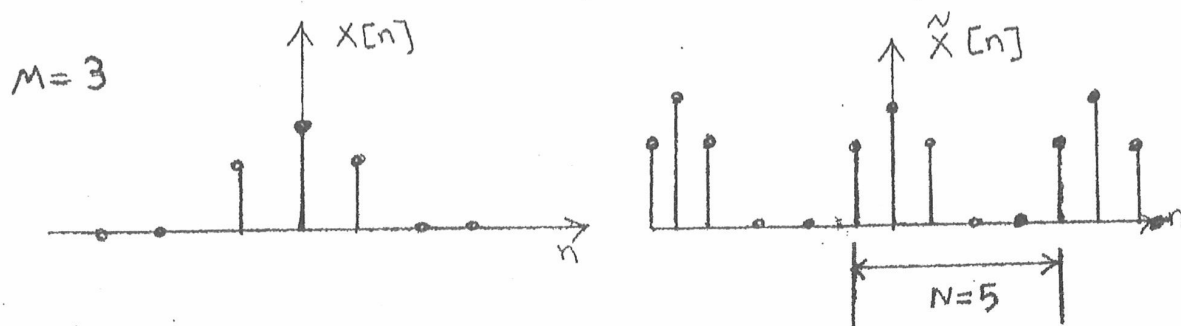
where $X[n]$ is a periodic sequence with period (N), and the spectral coefficients $\tilde{X}[K]$ are evaluated for $(0 \leq K \leq N-1)$, $\tilde{X}[K]$ is also periodic with a period (N). The inverse of above process which allows us to recover the signal from its spectrum, is given by :-

$$\tilde{X}[n] = \frac{1}{N} \sum_{K=0}^{N-1} \tilde{X}[K] e^{j2\pi Kn/N} \quad \dots\dots\dots (3.2)$$

Truly periodic signals are rarely encountered in practical DSP. Non-periodic (aperiodic) signals with a finite number of non zero sample values are the more common.

Let $x[n]$ be aperiodic signal with duration containing (M) samples. We will consider it is periodic signal of period (N) where $N \geq M$ then :-

$$\tilde{X}[n] = \begin{cases} x[n] & \text{for } 0 \leq n \leq M-1 \\ 0 & \text{for } M \leq n \leq N-1 \end{cases}$$



The sequence $\tilde{X}[n]$ is called the "periodic extension" of $X[n]$. We are free to choose the value of N , but we must be careful not to be too small, because, if $N < M$, an overlap then occurs when periodic extension is formed.

Summarizing all the above, we arrive at the desired DFT relationship

$$X[k] = \sum_{n=0}^{N-1} X[n] e^{-j2\pi n k / N} \quad \text{for } 0 \leq k \leq N-1$$

$$= \sum_{n=0}^{N-1} X[n] \left[\cos\left(\frac{2\pi n k}{N}\right) - j \sin\left(\frac{2\pi n k}{N}\right) \right]$$

$$X[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi n k / N} \Rightarrow \text{to recover the signal from its spectrum}$$

Sometimes we assume $e^{-j2\pi / N} = W_N$, thus $e^{-j2\pi k n / N} = W_N^{nk}$

The two equations form the basis of the computer algorithms that evaluate the DFT.

Fourier Transform (FT)

① DTFT (Discrete Time Fourier Transform)

② DFT (Discrete Fourier Transform)

③ FFT (Fast Fourier Transform)

① DTFT :-

$$(*) X(e^{j\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$(*) \text{ Energy spectral density} = |X(e^{j\omega})|^2$$

$$(*) e^{\pm jx} = \cos x \pm j \sin x$$

Ex 1

find the DTFT for $x(n) = (0.5)^n \cdot u(n)$

Sol :-

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5)^n \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} ((0.5) \cdot (e^{-j\omega}))^n$$

$$u(n) \Big|_{n=0}^{\infty}$$

$$= \frac{1}{1 - 0.5 e^{-j\omega}}$$

$$= \frac{1}{1 - 0.5 (\cos \omega - j \sin \omega)}$$

$$X(e^{j\omega}) = \frac{1}{(1 - 0.5 \cos \omega) + (j 0.5 \sin \omega)}$$

$\uparrow$ real part $\uparrow$ imag

To compute the Energy = $|X(e^{j\omega})|^2$ is :-

$$|X(e^{j\omega})| = \text{magnitude} = \frac{1}{\sqrt{(1 - 0.5 \cos \omega)^2 + (0.5 \sin \omega)^2}}$$

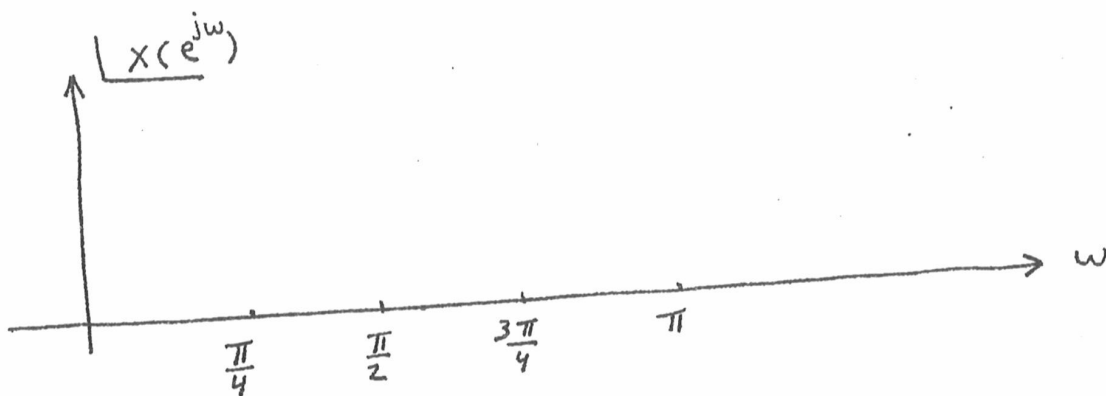
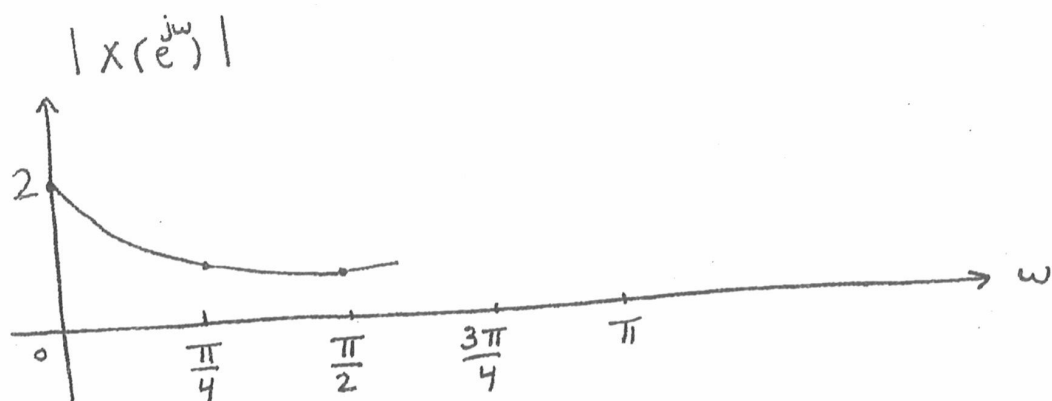
$$\angle X(e^{j\omega}) = \text{phase} = \underbrace{0}_{\tan^{-1} \frac{0}{1}} - \tan^{-1} \frac{0.5 \sin \omega}{1 - 0.5 \cos \omega}$$

$$\textcircled{*} \quad \tan^{-1} \frac{+no}{0} = +90^\circ$$

$$\tan^{-1} \frac{-no}{0} = -90^\circ$$

$$\textcircled{*} \quad \tan^{-1} \frac{0}{+no} = 0$$

$$\tan^{-1} \frac{0}{-no} = 180^\circ$$



ω	$ X(e^{j\omega}) $	$\angle X(e^{j\omega})$
0		
$\frac{\pi}{4}$		
$\frac{\pi}{2}$		
$\frac{3\pi}{4}$		

H.w: Find the energy density for $x[n] = u[n]$, using DTFT?

2] DFT Discrete Fourier Transform (DFT) Exo

$$\begin{aligned} \textcircled{*} X[k] &= \sum_{n=0}^{N-1} x[n] \cdot e^{-j \frac{2\pi kn}{N}} \\ &= \sum_{n=0}^{N-1} x[n] \cdot W_N^{nk} \end{aligned} \quad \left. \vphantom{\sum_{n=0}^{N-1}} \right\} \text{DFT}$$

$$\begin{aligned} \textcircled{*} x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot e^{+j \frac{2\pi kn}{N}} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot W_N^{-nk} \end{aligned} \quad \left. \vphantom{\sum_{k=0}^{N-1}} \right\} \text{(IDFT)}$$

Ex_o Consider the sequence $\{1, 4, 3, 8\}$ of the signal $x[n]$, find the DFT or $X[k]$?

Sol_o

$$N = 4 \quad n = k = 0 \longrightarrow 3$$

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j \frac{2\pi nk}{N}}$$

$$X[k] = \sum_{n=0}^3 x[n] \cdot e^{-j \frac{2\pi nk}{4}}$$

$$X[k] = \sum_{n=0}^3 x[n] \cdot e^{-\frac{\pi nk}{2} j}$$

$$X[0] = \sum_{n=0}^3 x[n] \cdot \cancel{e^0}$$

$$= x[0] + x[1] + x[2] + x[3]$$

$$= 1 + 4 + 3 + 8 = \boxed{16}$$

$$X[1] = \sum_{n=0}^3 x[n] e^{-j \frac{n\pi}{2}}$$

$$= x[0] + x[1] e^{-j \frac{\pi}{2}} + x[2] e^{-j\pi} + x[3] e^{-j \frac{3\pi}{2}}$$

$$= 1 + 4 \left(\overset{0}{\cancel{\cos \frac{\pi}{2}}} - \overset{1}{\cancel{\sin \frac{\pi}{2}}} j \right) + 3 \left(\overset{-1}{\cancel{\cos \pi}} - \overset{0}{\cancel{j \sin \pi}} \right) + 8 \left(\overset{0}{\cancel{\cos \frac{3\pi}{2}}} - \overset{-1}{\cancel{j \sin \frac{3\pi}{2}}} \right)$$

$$= 1 - 4j - 3 + 8j = \boxed{-2 + 4j}$$

$$X[2] = x[0] + x[1] e^{-j\pi} + x[2] e^{-2\pi j} + x[3] e^{-j3\pi}$$

$$= 1 + 4 \left(\overset{-1}{\cancel{\cos \pi}} - \overset{0}{\cancel{j \sin \pi}} \right) + 3 \left(\overset{1}{\cancel{\cos 2\pi}} - \overset{0}{\cancel{j \sin 2\pi}} \right) + 8 \left(\overset{-1}{\cancel{\cos 3\pi}} - \overset{0}{\cancel{j \sin 3\pi}} \right)$$

$$= 1 - 4 + 3 - 8 = \boxed{-8}$$

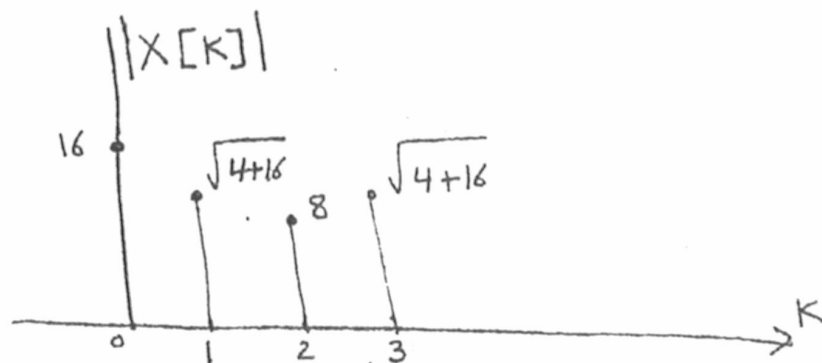
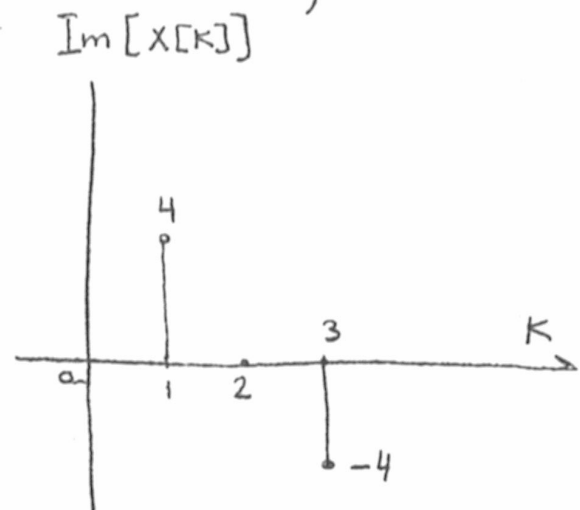
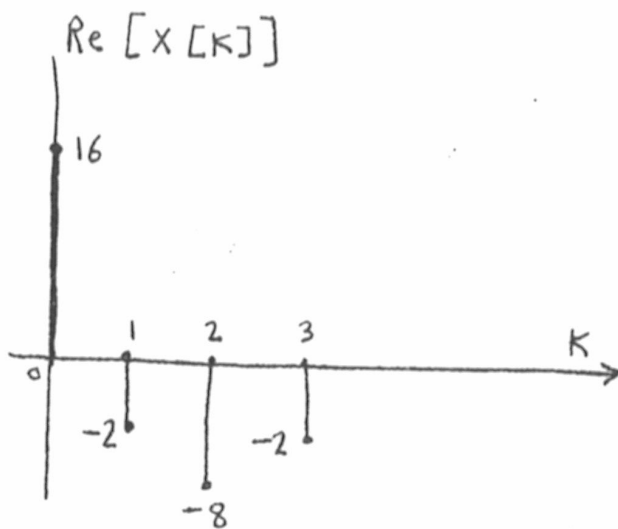
$$X[3] = \sum_{n=0}^3 X[n] \cdot e^{-jn\pi/2}$$

$$X[3] = X[0] + X[1]e^{-j\frac{3\pi}{2}} + X[2]e^{-j3\pi} + X[3]e^{j\frac{9\pi}{2}}$$

$$= 1 + 4(\cos \frac{3\pi}{2} - \sin \frac{3\pi}{2}j) + 3(\cos 3\pi - \sin 3\pi j) + 8(\cos \frac{9\pi}{2} - j \sin \frac{9\pi}{2})$$

$$= -2 - j4$$

$$X[K] = \{ 16, -2 + 4j, -8, -2 - 4j \}$$



Exo: Compute the N -point DFT, where the $N=3$ For the sequence

$$h[n] = \begin{cases} \frac{1}{3} & 0 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Sol: $\Rightarrow$

$$h[n] = \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\}$$

$$H[K] = \sum_{n=0}^{N-1} h[n] \cdot e^{-j \frac{2\pi n K}{N}}$$

$$H[K] = \sum_{n=0}^2 \frac{1}{3} \cdot e^{-j \frac{2\pi n K}{3}}$$

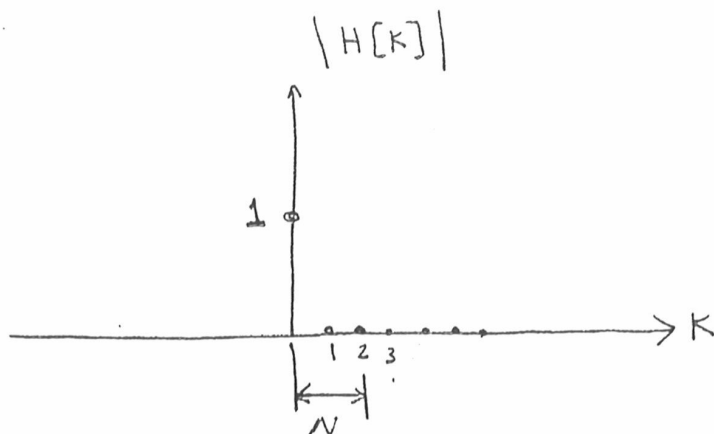
$$H[K] = \frac{1}{3} + \frac{1}{3} e^{-j \frac{2\pi K}{3}} + \frac{1}{3} e^{-j \frac{4\pi K}{3}}$$

$$H[0] = \frac{1}{3} + \frac{1}{3} e^0 + \frac{1}{3} e^0 = \boxed{1}$$

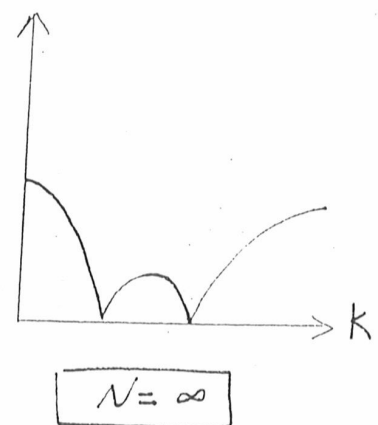
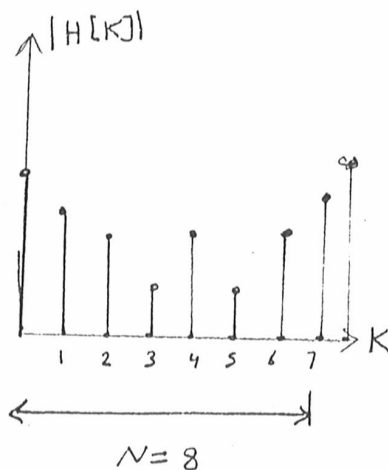
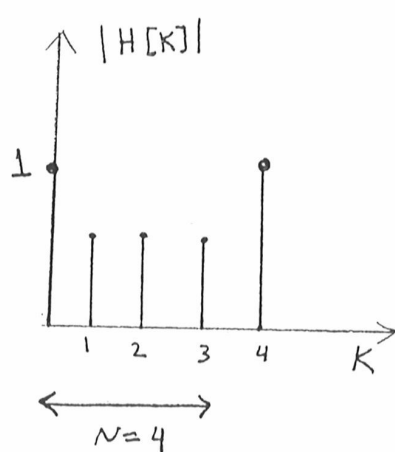
$$H[1] = \frac{1}{3} + \frac{1}{3} e^{-j \frac{2\pi}{3}} + \frac{1}{3} e^{-j \frac{4\pi}{3}}$$

$$= \frac{1}{3} + \frac{1}{3} \left(\overset{-0.5}{\cos \frac{2\pi}{3}} - j \overset{-0.86j}{\sin \frac{2\pi}{3}} \right) + \frac{1}{3} \left(\overset{-0.5}{\cos \frac{4\pi}{3}} - j \overset{+0.86j}{\sin \frac{4\pi}{3}} \right)$$

$$= 0 + \frac{1}{3} e^{-j \frac{4\pi}{3}} + \frac{1}{3} e^{-j \frac{8\pi}{3}} = 0$$



Note that increasing " N " will increase number of samples in $H[k]$



H.w Compute DFT for the infinite-duration sequence given by:

$$h[n] = a^n u[n]$$

Note: The following table shows DFT pairs of some important sequences.

Table (3.1)

$x[n]$	$X[K]$
$\delta[n]$	1
$\delta[n-a]$	$e^{-j \frac{2\pi K a}{N}}$
$a^n u[n]$	$\frac{1}{1 - a e^{-j 2\pi K / N}}$
$u[n]$	$\frac{1}{1 - e^{-j 2\pi K / N}}$
$a^n e^{j 2\pi K n / N} u[n]$	$\frac{1}{1 - a e^{-j 2\pi K / N}}$

Ex: 0

prove that DFT of ① $\delta[n] = 1$

② $\delta[n-n_0] = W_N^{Kn_0}$

Sol: $X[K] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi K n}{N}}$

① since $x[n] = \delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow X[K] = 1 \cdot e^0 + 0 + 0 + \dots = 1$

② $X[K] = \sum_{n=0}^{N-1} \delta[n-n_0] e^{-j \frac{2\pi K n}{N}} = e^{-j \frac{2\pi K n_0}{N}} = W_N^{Kn_0}$

3.2.1 Properties of DFT

1- Linearity :

$$A x_1[n] + B x_2[n] \Leftrightarrow A X_1[k] + B X_2[k]$$

2- Convolution :

$$x_1[n] \otimes x_2[n] \Leftrightarrow X_1[k] \cdot X_2[k]$$

3- Modulation :

$$x_1[n] \cdot x_2[n] \Leftrightarrow X_1[k] \otimes X_2[k]$$

4- Periodicity :

$$X[k] = X[k+N]$$

5- Time shifting :

$$x[n-a] \Rightarrow X[k] e^{-j \frac{2\pi k}{N} a}$$

6- Frequency Shifting :

$$x[n] e^{j \frac{2\pi n a}{N}} \Leftrightarrow X[(k-a)N] = \begin{cases} X[k-a] & a \leq k \leq N-1 \\ X[k-a+N] & 0 \leq k \leq a \end{cases}$$

7- Parseval's Relation :-

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

Ex:- compute the energy in the sequence

$$x[n] = \frac{1}{3} \delta[n] + \frac{1}{3} \delta[n-1] + \frac{1}{3} \delta[n-2]$$

Solu:-
The energy in the time domain = $\sum_{n=0}^2 |x[n]|^2 \Rightarrow \frac{1}{3}$

The energy in the freq. domain = $\frac{1}{N} \sum_{k=0}^2 |X[k]|^2 \Rightarrow \frac{1}{3}$

Ex:- Compute the IDFT for $H[k] = \frac{1 + 2 \cos(\frac{2\pi k}{N})}{5}$

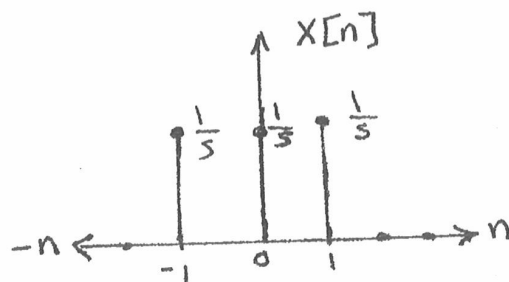
Solu:-

$$\text{we have } h[n] = \frac{1}{N} \sum_{k=0}^{N-1} H[k] e^{j \frac{2\pi k n}{N}}$$

or we can directly find the IDFT from table.
since

$$\begin{aligned} X[k] &= \frac{1}{5} + \frac{2}{5} \left(\frac{e^{j \frac{2\pi k}{N}} + e^{-j \frac{2\pi k}{N}}}{2} \right) \\ &= \frac{1}{5} + \frac{1}{5} e^{j \frac{2\pi k}{N}} + \frac{1}{5} e^{-j \frac{2\pi k}{N}} \end{aligned}$$

$$\Rightarrow x[n] = \frac{1}{5} \delta[n] + \frac{1}{5} \delta[n+1] + \frac{1}{5} \delta[n-1]$$



3 Fast Fourier Transform (FFT)

FFT is used to increase the computational speed of the DFT. There are two algorithms for FFT:-

- 1- Decimation in time algorithm (DIT)
- 2- Decimation in frequency algorithm (DIF)

We are going to explain DIT-FFT only.

$$X[K] = \sum_{n=0}^{N-1} X[n] e^{-j2\pi nK/N} \quad \text{for } 0 \leq K \leq N-1$$

$$= \sum_{n=0}^{N-1} X[n] W_N^{nK}$$

$$= \sum_{\substack{n=0 \\ \text{even}}}^{N-1} X[n] W_N^{nK} + \sum_{\substack{n=0 \\ \text{odd}}}^{N-1} X[n] W_N^{nK}$$

Letting $n = 2r$ for even part
 $n = 2r+1$ for odd part

$$\Rightarrow X[K] = \sum_{r=0}^{\frac{N}{2}-1} X[2r] W_{N/2}^{rK} + W_N^K \sum_{r=0}^{\frac{N}{2}-1} X[2r+1] W_{N/2}^{rK}$$

⊗ (Mathematical) ⊗

we can demonstrate the decimation-in-time process quite easily in the case of a short transform, for example

$N=8$. The signal $X[n]$ is defined for:-

$$n = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

we form two subsequences of the even-numbered

odd-numbered points (radix-2 FFT):-

$$n = \{0, 2, 4, 6\} \text{ and } \{1, 3, 5, 7\}$$

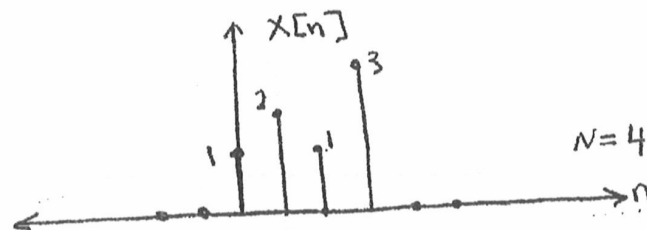
use of this modified module leads to a reduction in the total number of multiplications by 50%.

Now, each subsequence can be further decimated giving:-

$$n = \{0, 4\} \quad \{2, 6\} \quad \{1, 5\} \quad \{3, 7\}$$

And the later is called radix-4 FFT.

Ex:- Determine $X[K]$ for the input sequence shown below using FFT algorithm with $N/2$ point. Given $M=4$, $N=4$.



$$\text{Sol:- } X[K] = \sum_{r=0}^{\frac{N}{2}-1} X[2r] W_{N/2}^{rK} + W_N^K \sum_{r=0}^{\frac{N}{2}-1} X[2r+1] W_{N/2}^{rK}$$

$$= X[0] + X[2] W_2^K + W_4^K (X[1] + X[3] W_2^K)$$

$$= 1 + W_2^K + W_4^K (2 + 3 W_2^K)$$

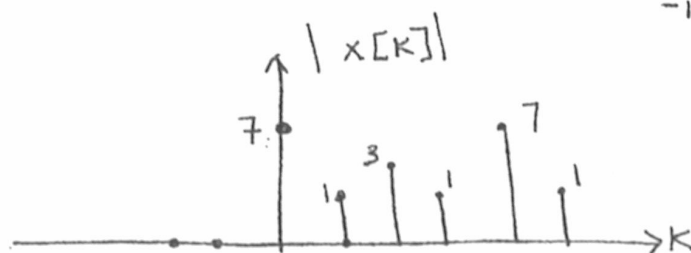
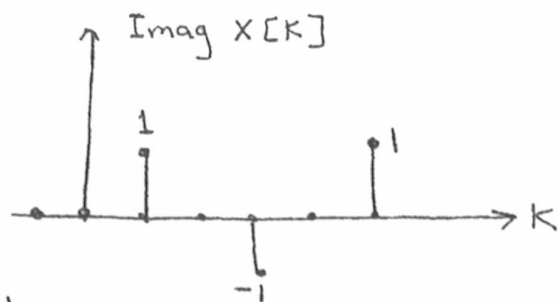
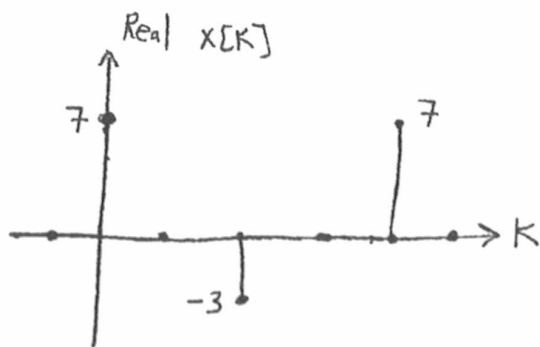
$$X[K] = 1 + e^{-j\pi K} + 2 e^{-j\frac{\pi K}{2}} + 3 e^{-j\frac{3\pi K}{2}} =$$

$$X[0] = 1 + 1 + 2 + 3 = 7$$

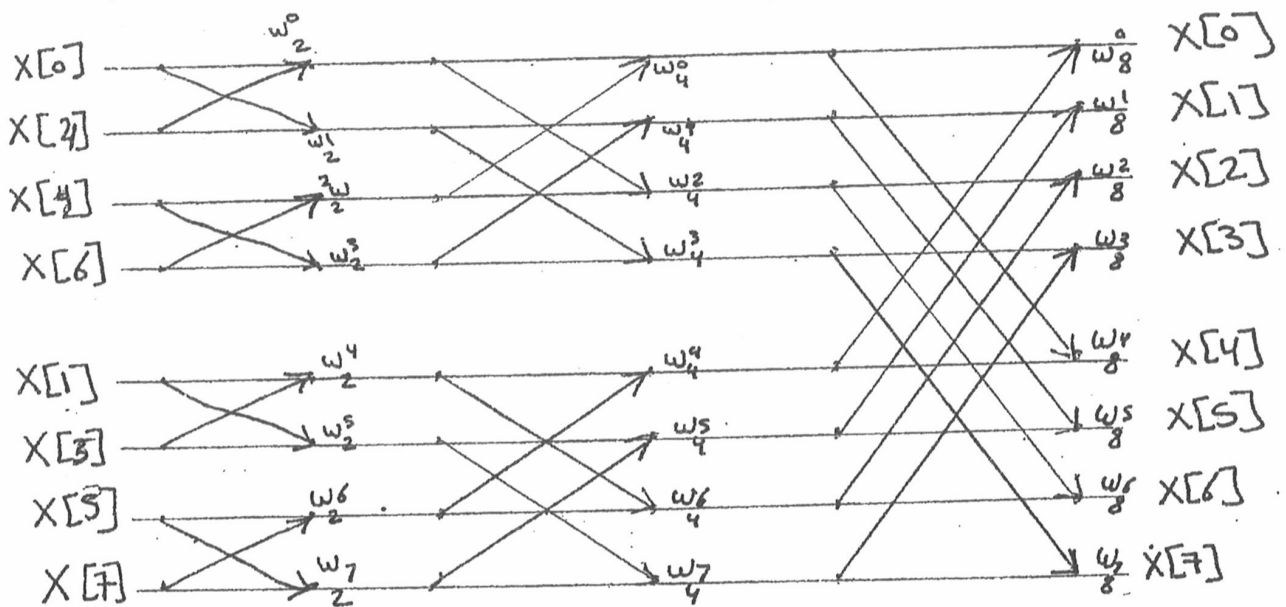
$$X[1] = 1 + e^{-j\pi} + 2 e^{-j\frac{\pi}{2}} + 3 e^{-j\frac{3\pi}{2}} = j$$

$$X[2] = 1 + e^{-j2\pi} + 2e^{-j\pi} + 3e^{-j3\pi} = -3$$

$$X[3] = 1 + e^{-j3\pi} + 2e^{-j\frac{3\pi}{2}} + 3e^{-j\frac{9\pi}{2}} = -j$$



FFT (Graphical or Butterfly)



$$w_2^0 = 1$$

$$w_2^1 = -1$$

$$w_2^2 = 1$$

$$w_2^3 = -1$$

$$w_2^4 = 1$$

$$w_2^5 = -1$$

$$w_2^6 = 1$$

$$w_2^7 = -1$$

$$w_4^0 = 1$$

$$w_4^1 = -j$$

$$w_4^2 = -1$$

$$w_4^3 = j$$

$$w_4^4 = 1$$

$$w_4^5 = -j$$

$$w_4^6 = -1$$

$$w_4^7 = j$$

$$w_8^0 = 1$$

$$w_8^1 = \frac{\sqrt{2}}{2} (1-j)$$

$$w_8^2 = -j$$

$$w_8^3 = -\frac{\sqrt{2}}{2} (1+j)$$

$$w_8^4 = -1$$

$$w_8^5 = \frac{\sqrt{2}}{2} (1-j)$$

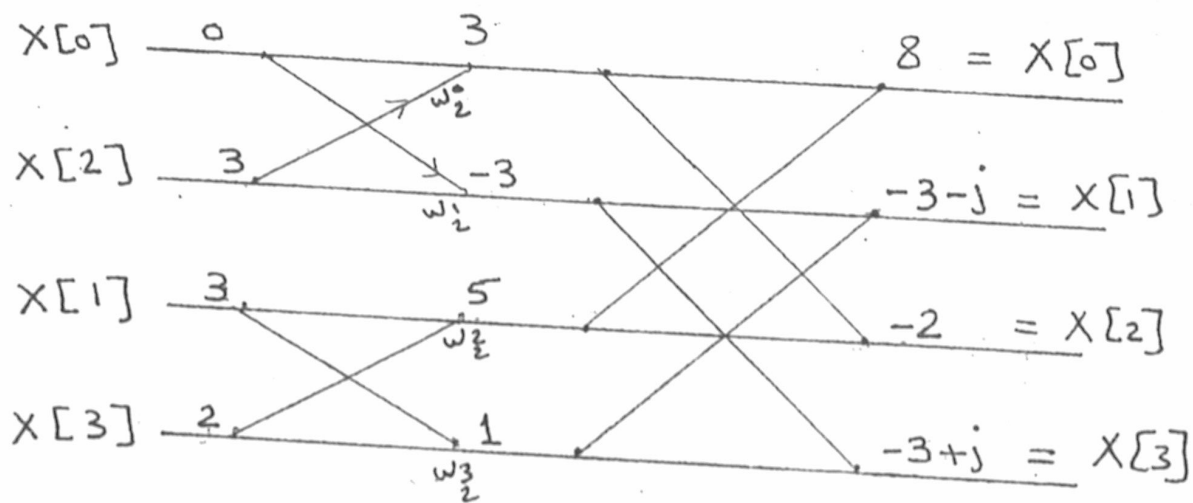
$$w_8^6 = j$$

$$w_8^7 = \frac{\sqrt{2}}{2} (1+j)$$

Ex:- Find the FFT of $x[n] = [0 \ 3 \ 3 \ 2]$ using Graphical method.

Soln:-

$$\left. \begin{array}{l} X[0] = 0 \\ X[1] = 3 \\ X[2] = 3 \\ X[3] = 2 \end{array} \right\} \text{Reverse} \quad \begin{array}{l} X[0] \\ X[2] \\ X[1] \\ X[3] \end{array}$$



H.w) Find the FFT of $x[n] = [1 \ 2 \ 1]$ using Graphical method.

Ex:- Find the FFT of $x[n] = [2, -2, -2, -2, 2, 2, 2, -2]$

Sol:-

$$X[0] = 2$$

$$X[1] = -2$$

$$X[2] = -2$$

$$X[3] = -2$$

$$X[4] = 2$$

$$X[5] = 2$$

$$X[6] = 2$$

$$X[7] = -2$$

(Reverse the order)

$$X[0]$$

$$X[4]$$

$$X[2]$$

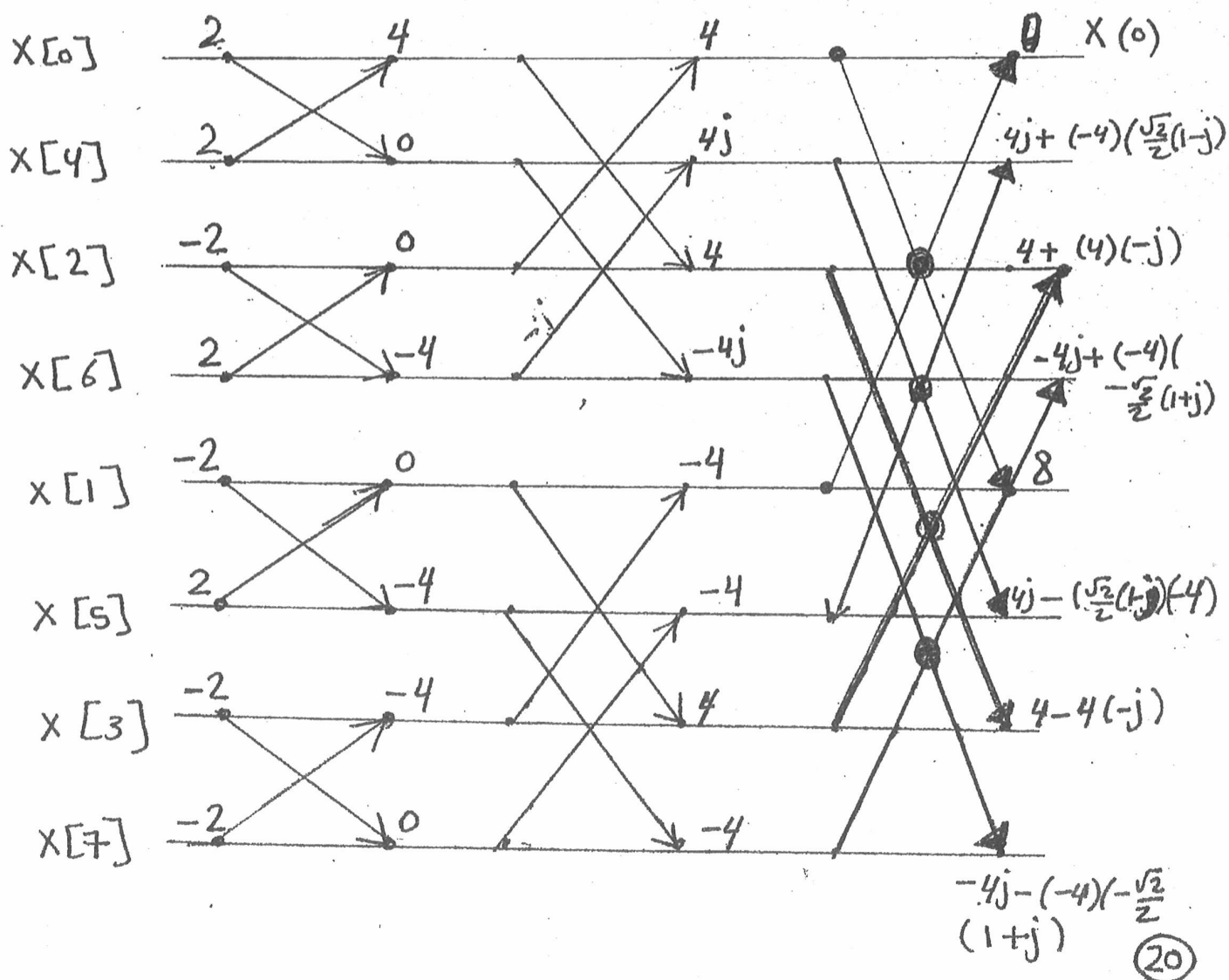
$$X[6]$$

$$X[1]$$

$$X[5]$$

$$X[3]$$

$$X[7]$$



notes :-

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no of operation in DFT =  $N^2$

no of operation in FFT =  $N \log_2 N$

Ex:-

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Let $N = 16$

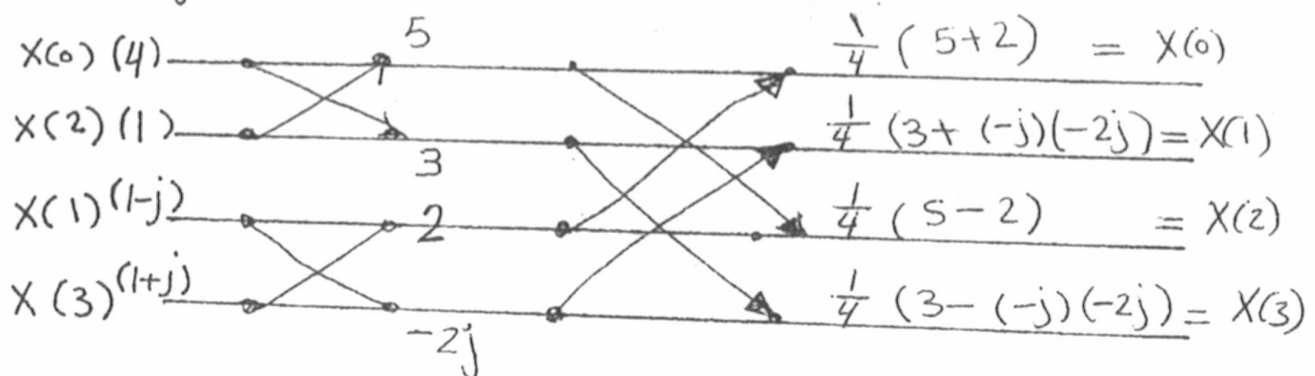
no of operation in DFT = $N^2 = (16)^2 = 256$

no of operation in FFT = $N \log_2 N = 16 \log_2 2^4 = 16 \times 4$
 $= 64$

This leads to the FFT is faster in computation from DFT.

Ex:- $x[n] = [4, 1-j, 1, 1+j]$ find the IFFT?

Sol:-



Z-Transform :-

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The z-transformation does for analysis and design of digital systems what the Laplace transformation does for analysis and design of analog systems.

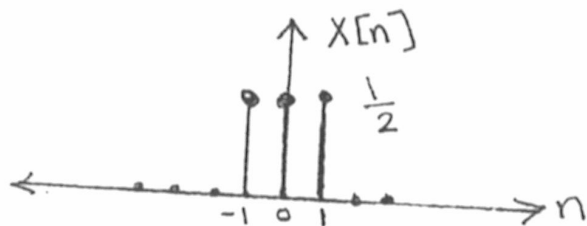
The z-transform of a digital signal $x[n]$ is

$$X[z] = \sum_{n=0}^{\infty} x[n] z^{-n} \quad \text{one sided z-transform}$$

The z-transform and the DFT are closely related

$$X[z] = X[k] \Big|_{e^{j2\pi k/N} = z}$$

Ex:- Find the z-transform of the signal shown below:



Sol :-

$$\begin{aligned} X(z) &= \sum_{n=-1}^1 x[n] z^{-n} \\ &= \frac{1}{2} z + \frac{1}{2} z^0 + \frac{1}{2} z^{-1} \end{aligned}$$

①

Ex:- Find the z-transform of exponentially signal given by:- $X[n] = a^n U[n]$

Sol:-

$$X[z] = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$X[z] = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a} \quad \text{for } |a z^{-1}| < 1$$

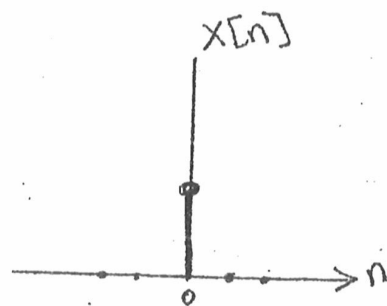
Ex:- $X[n] = \delta[n]$, find $X[z]$

Sol:-

$$X[z] = \sum_{n=-\infty}^{\infty} X[n] \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n) \cdot z^{-n}$$

$$= 1 \cdot z^{-0} = 1$$



(*) note

$$\textcircled{1} X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} \quad \left[\begin{array}{l} \text{Two-sided} \\ \text{z-transform} \end{array} \right]$$

$$\textcircled{2} \cos w = \frac{e^{jw} + e^{-jw}}{2}$$

$$\textcircled{3} \sin w = \frac{e^{jw} - e^{-jw}}{2j}$$

1- linearity :o

$$A x_1[n] + B x_2[n] \Leftrightarrow A x_1[z] + B x_2[z]$$

2- Time - shifting :o

$$x[n-a] \Leftrightarrow X(z) z^{-a}$$

3- Convolution :o

$$x_1[n] \otimes x_2[n] \Leftrightarrow X_1[z] \cdot X_2[z]$$

4 - Differentiation :o

$$n x[n] \Leftrightarrow (-z) \frac{dX(z)}{dz}$$

5 - Multiplication by an exponential sequence :o

$$a^n x[n] \Rightarrow X\left(\frac{z}{a}\right)$$

z-transform pairs of some important signals:-

$x[n]$

$X[z]$

$$\textcircled{1} \quad \delta[n] \xrightarrow{z.T} 1$$

$$\textcircled{2} \quad \delta[n-a] \xrightarrow{z.T} z^{-a} = \frac{1}{z^a}$$

$$\textcircled{3} \quad u[n] \xrightarrow{z.T} \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$\textcircled{4} \quad a^n u[n] \xrightarrow{z.T} \frac{z}{(z-a)}$$

$$\textcircled{5} \quad n u[n] \xrightarrow{z.T} \frac{z}{(z-1)^2}$$

$$\textcircled{6} \quad x[n-a] \xrightarrow{z.T} z^{-a} X(z)$$

Ex:- Find the z-transform for $r[n]$:-

Sol:-

$$\begin{aligned} r[n] = n u[n] &\xrightarrow{z.T} -z \frac{dX(z)}{dz} \\ &\longrightarrow -z \frac{d}{dz} \left(\frac{z}{z-1} \right) \\ &\longrightarrow -z \cdot \frac{(z-1) - (z-1)}{(z-1)^2} \\ &\longrightarrow \frac{z}{(z-1)^2} \end{aligned}$$

(4)

Ex:0 (linearity)

$$3\delta(n) + (0.3)^n \xrightarrow{z.T} 3 + \frac{z}{z-0.3}$$

Ex:0 (Time shifting)

$$\begin{aligned} u[n-4] &\xrightarrow{z.T} z^{-4} \cdot \frac{z}{z-1} \\ &= \frac{z^{-3}}{z-1} \end{aligned}$$

Ex:0 (multiplication by a^n)

$$y[n](5)^n \longrightarrow n u[n] \cdot (5)^n$$

$$n u[n] \xrightarrow{z.T} \frac{z}{(z-1)^2}$$

$$n u[n](5)^n \xrightarrow{z.T} \frac{\frac{z}{5}}{\left(\frac{z}{5}-1\right)^2} = 5 \frac{z}{(z-5)^2}$$

Ex:0 (Differentiation)

$$n \cdot 3^n \xrightarrow{z.T} -z \frac{d}{dz} \left(\frac{z}{z-3} \right)$$

$$\frac{-z(z-3-z)}{(z-3)^2}$$

$$= 3 \frac{z}{(z-3)^2}$$

Sol: =

$$\text{let } x_1[n] = (-2)^n u[n] \xrightarrow{z.T} \frac{z}{z+2}$$

$$\text{let } x_2[n] = u[n] \xrightarrow{z.T} \frac{z}{z-1}$$

$$Z[x_1[n] \otimes x_2[n]] = \frac{z}{z+2} \cdot \frac{z}{z-1}$$

Ex: = Find the z-transform for $n u[n] \otimes \delta[n]$

Sol: =

$$\text{let } x_1[n] = n u[n] \xrightarrow{z.T} \frac{z}{(z-1)^2}$$

$$\text{let } x_2[n] = \delta[n] \xrightarrow{z.T} 1$$

$$\begin{aligned} Z[x_1[n] \otimes x_2[n]] &= X_1[z] \cdot X_2[z] \\ &= \frac{z}{(z-1)^2} \end{aligned}$$

(K)

Inverse Z-Transforms

Two simple methods for the inverse transform computation are reviewed in the next two examples.

① Partial Fractions

Ex: A signal has the z-transform $X(z) = \frac{1}{z(z-1)(2z-1)}$
use the method of partial fractions to recover the signal $x[n]$.

soln:

$$X(z) = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{2z-1}$$

$$A = \lim_{z \rightarrow 0} \frac{1}{(z-1)(2z-1)} = 1$$

$$B = \lim_{z \rightarrow 1} \frac{1}{z(2z-1)} = 1$$

$$C = \lim_{z \rightarrow 0.5} \frac{1}{z(z-1)} = -4$$

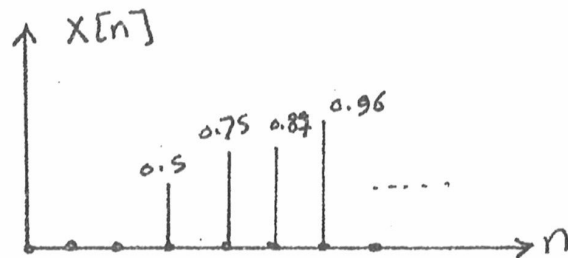
$$X(z) = \frac{1}{z} + \frac{1}{z-1} + \frac{-4}{2z-1}$$

$$= \frac{1}{z} + \frac{1}{z-1} - \frac{2}{z-0.5} \} \times [z^{-1}]$$

$$X(z) = z^{-1} + z^{-1} \frac{z}{z-1} - z^{-1}(2) \frac{z}{z-0.5}$$

Referring to table $\Rightarrow$

$$X[n] = S[n-1] + U[n-1] - 2 \cdot 0.5^{n-1} U[n-1]$$



② Long Division

Ex: solve the previous example using long division the numerator by the denominator.

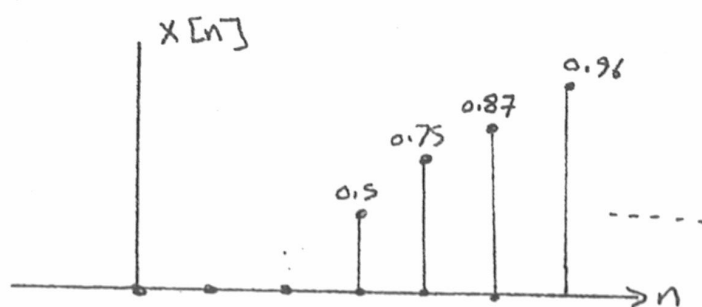
$$X(z) = \frac{1}{z(z-1)(2z-1)}$$

$$= \frac{1}{2z^3 - 3z^2 + z}$$

$$\begin{array}{r} 2z^3 - 3z^2 + z \overline{) 1} \\ \underline{0.5z^{-3}} \\ 1 \\ \underline{1 - 1.5z^{-1} + 0.5z^{-2}} \\ 1.5z^{-1} - 0.5z^{-2} \\ \underline{1.5z^{-1} - 2.25z^{-2} + 0.75z^{-3}} \\ 1.75z^{-2} - 0.75z^{-3} \\ \vdots \end{array}$$

$$X[z] = 0.5 z^{-3} + 0.75 z^{-4} + 0.875 z^{-5} + \dots$$

$$X[n] = 0.5 \delta(n-3) + 0.75 \delta(n-4) + 0.875 \delta(n-5) + \dots$$



Exgo find $h[n]$ for

$$H(z) = \frac{1 + z^{-1}}{1 + z^{-1} + 2z^{-2} + 2z^{-3}}$$

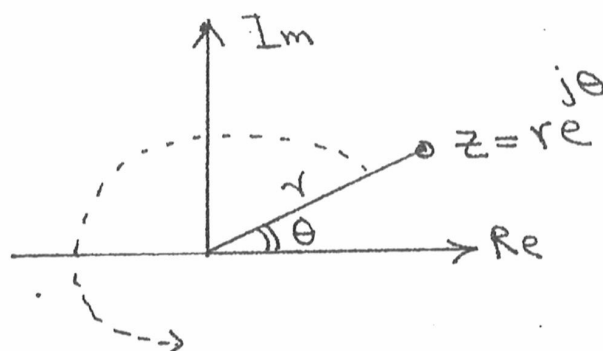
$$\begin{array}{r} \frac{1 + z^{-1}}{1 + z^{-1} + 2z^{-2} + 2z^{-3}} \left[\frac{1 - 2z^{-2} + 4z^{-4}}{1 + z^{-1}} \right] \\ \hline 1 + z^{-1} + 2z^{-2} + 2z^{-3} \\ \hline -2z^{-2} - 2z^{-3} \\ -2z^{-2} - 2z^{-3} - 4z^{-4} - 4z^{-5} \\ \hline 4z^{-4} + 4z^{-5} \\ 4z^{-4} + 4z^{-5} + 8z^{-6} + 8z^{-7} \\ \hline -8z^{-6} - 8z^{-7} \end{array}$$

$$X(z) = 1 - 2z^{-2} + 4z^{-4} \dots$$

$$X[n] = \delta(n) - 2\delta(n-2) + 4\delta(n-4) \dots$$

z-plane

The independent variable (z) is a complex variable. Values of (z) can be associated with points in a plane called the z -plane. The z -plane is an important graphical tool in the theory and application of the z -transformation.



Stability Determination Based z-Transform

A digital signal or an LTI system can always be described using z -transform as the ratio

$$X(z) = \frac{A(z)}{B(z)} = \frac{K(z-z_1)(z-z_2)(z-z_3)\dots}{(z-p_1)(z-p_2)\dots}$$

The constants $z_1, z_2, z_3, \dots$ are called the "zeros" of $X[z]$, because they are the values of (z) for which $X[z]$ is zero. Conversely $p_1, p_2, \dots$ are known as the poles of $X[z]$. The poles and zeros are either real, or occur in complex conjugate pairs.

"The digital system is stable, if and only if, all the poles of the system lie inside the unit circle

in the z -plane". K = the system gain.

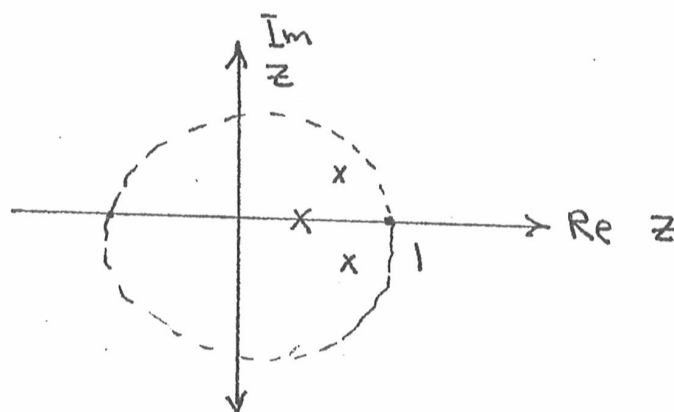
Ex 80

check the stability of the system given by:

$$H(z) = \frac{K(z-1)^2}{(z-0.3)(z^2 - z + 0.5)}$$

Sol 80

$$H(z) = \frac{K(z-1)^2}{(z-0.3)(z-0.5+j0.5)(z-0.5-j0.5)}$$

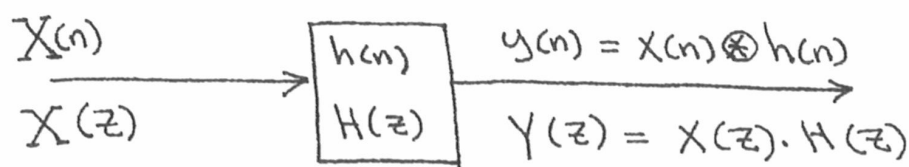


Since all the poles lie inside the unit circle
∴ the system is stable.

Evaluation of LTI System Response Using z -Transform

The output of LTI for input $x[n]$ can be obtained using z -transformation:

$$y[n] = \sum_{k=-\infty}^{\infty} Y(z) = \sum_{k=-\infty}^{\infty} X[z] \cdot H(z)$$



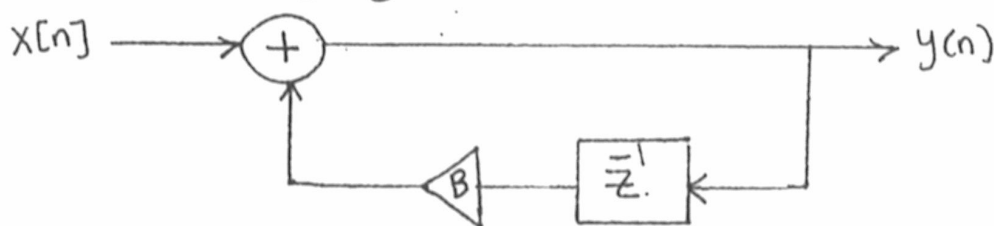
where $h[n]$ = impulse response of the system $= y[n] \big|_{x[n] = \delta[n]}$

$H(z)$ = transfer function of the

$$\text{system} = \mathcal{Z} h[n] \text{ or } = \frac{Y(z)}{X(z)}$$

Ex 8 =

Find the impulse response and the transfer function of the following system.



Soln: $y[n] = B y[n-1] + x[n]$

$$\Rightarrow Y(z) = B z^{-1} Y(z) + X(z)$$

for the impulse response $x[n] = \delta[n] \Rightarrow X(z) = 1$

$$\therefore Y(z) = B z^{-1} Y(z) + 1$$

$$Y(z) = \frac{1}{1 - B z^{-1}}$$

$$h[n] = y[n] \big|_{x[n] = \delta[n]} = \mathcal{Z}^{-1} \frac{1}{1 - B z^{-1}} = B^n u[n]$$

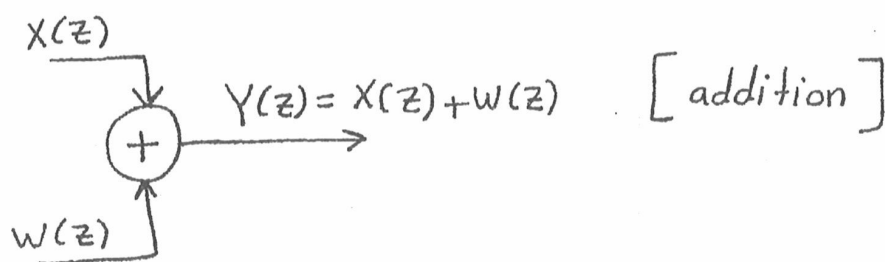
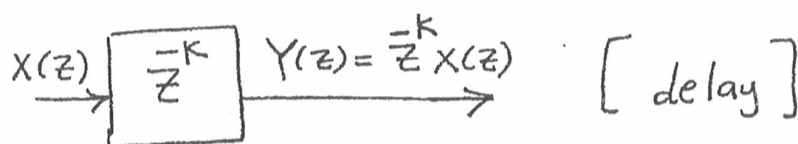
$$\text{T.F} = H(z) = \frac{Y(z)}{X(z)} = \mathcal{Z} h[n] = \frac{1}{1 - B z^{-1}}$$

Digital System Implementation from its Function

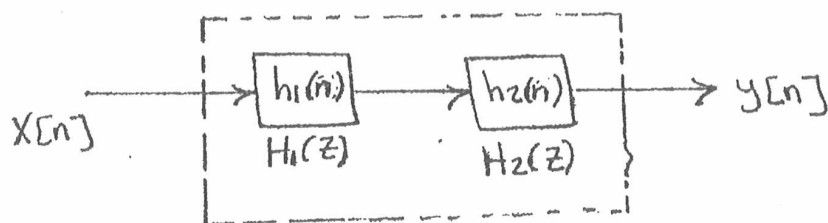
Since the z -transform is a linear transformation, the system implementation procedure is similar to that in the time domain. The most convenient form for system synthesis is the z -transform of the general difference equation given by

$$Y(z) = \sum_{k=1}^M a_k z^{-k} Y(z) + \sum_{k=0}^b b_k z^{-k} X(z)$$

This equation can be implemented using the following symbols for elementary LTI system



Note



$$h[n] = h_1[n] \otimes h_2[n]$$

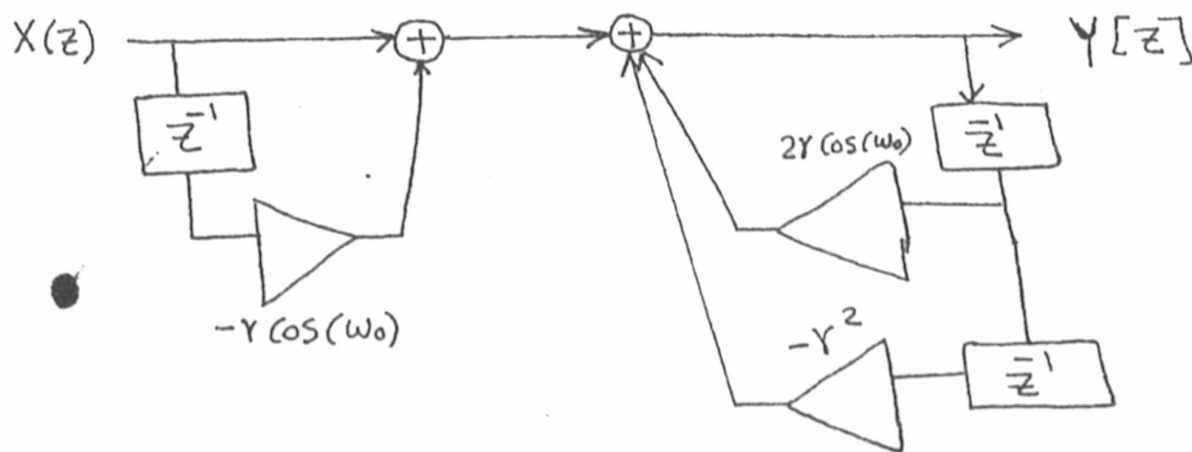
$$H[z] = H_1[z] \cdot H_2[z]$$

Ex: Implement the 2nd order recursive filter :-

$$y[n] = 2r \cos(\omega_0) y[n-1] - r^2 y[n-2] + x[n] - r \cos(\omega_0) x[n-1]$$

$$\text{Sol: } Y(z) = 2r \cos(\omega_0) \bar{z}^{-1} Y(z) - r^2 \bar{z}^{-2} Y(z) + X(z) - r \cos(\omega_0) \bar{z}^{-1} X(z)$$

The structure of the filter is shown below



Ex: Realize the digital system given by :-

$$H[z] = \frac{z(z+1)}{(z^2 - z + 0.5)}$$

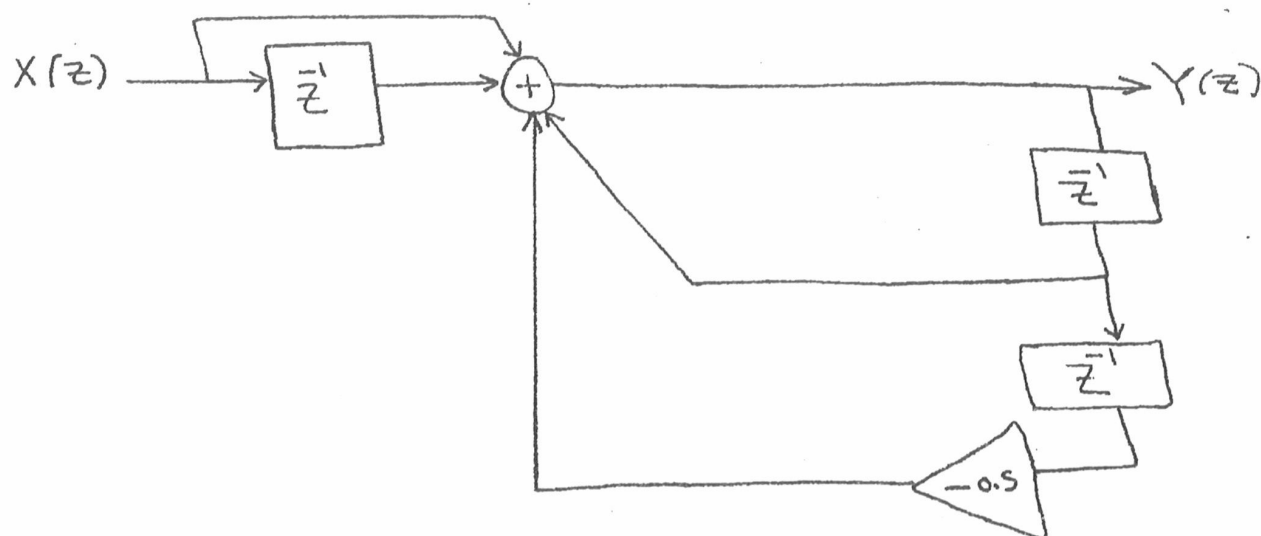
Sol :-

$$H[z] = \frac{Y[z]}{X[z]} = \frac{z(z+1)}{(z^2 - z + 0.5)} \quad \left\} * \frac{\bar{z}^{-2}}{\bar{z}^{-2}}\right.$$

$$\frac{Y[z]}{X[z]} = \frac{1 + \bar{z}^{-1}}{1 - \bar{z}^{-1} + 0.5 \bar{z}^{-2}}$$

$$Y[z] (1 - \bar{z}^{-1} + 0.5 \bar{z}^{-2}) = X[z] (1 + \bar{z}^{-1})$$

$$Y[z] = X[z] + \bar{z}^{-1} X[z] + \bar{z}^{-1} Y[z] - 0.5 \bar{z}^{-2} Y[z]$$

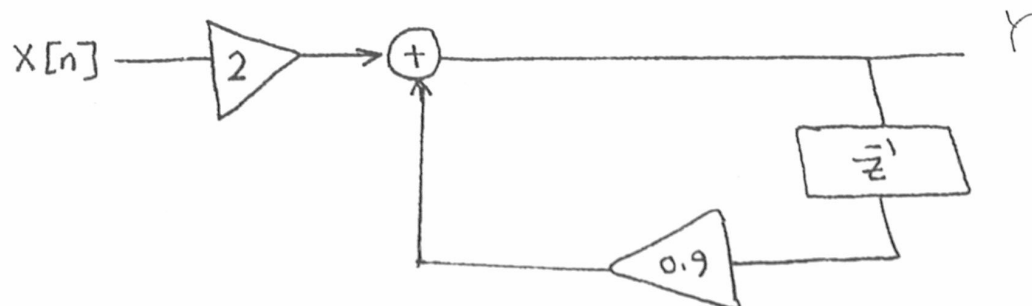


Frequency Response of LTI Processors

Most discrete time signals encountered in practice can be represented as a linear combination of a very large, may be infinite, number of sinusoidal discrete time signals of different angular frequencies. Thus, knowing the response of the LTI system to a single sinusoidal signal, we can determine its response to more complicated signals by making use of the superposition property of the system. Since a sinusoidal signal can be expressed in terms of an exponential signal, the response of the LTI system to an exponential input is of practical interest. This leads to the concept of frequency response, transform-domain representation of the LTI discrete-time system.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = H[k] \Big|_{z=e^{j\omega}} = H(z) \Big|_{z=e^{j\omega}}$$

Exo Find and sketch the frequency response of the system shown below.



Solu:-

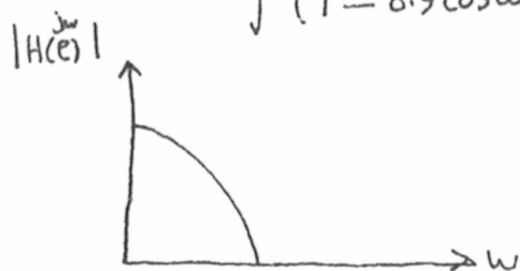
$$Y[z] = 2X[z] + 0.9 \bar{z}^{-1} Y[z]$$

$$H[z] = \frac{Y[z]}{X[z]} = \frac{2}{1 - 0.9 \bar{z}^{-1}}$$

$$\Rightarrow \text{Frequency response} = H(e^{j\omega}) = H(z) \Big|_{z=e^{j\omega}}$$

$$H(e^{j\omega}) = \frac{2}{1 - 0.9 \bar{z}^{-1}} = \frac{2}{1 - 0.9 \cos \omega + j 0.9 \sin \omega}$$

$$|H(e^{j\omega})| = \frac{2}{\sqrt{(1 - 0.9 \cos \omega)^2 + (0.9 \sin \omega)^2}}$$



Note:- The studying of the frequency response of the systems lead us to determine the behavior of the system with the frequencies.