

Frequency Response Analysis Using BODE PLOTS

Dr.Ziad Saeed Mohammed

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ENGINEERING TECHNICAL COLLEGE

Frequency Response

Types of frequency response plots

- **Polar plot (Nyquist plot):** real and imaginary part of open-loop system.
- **Bode plot:** magnitude and phase of open-loop system (begin with this one!!).
- **Nichols chart:** magnitude and phase of open-loop system in a different manner (not covered in the class).

BODE PLOTS

Bode Plot H.W. Bode (Bell Lab. 1939-42)

Frequency response $G(j\omega)$: the steady-state response of a system (forced response of an unstable system) with transfer function $G(s)$ to a sinusoid of unit amplitude and frequency ω is a sinusoid of amplitude $A(\omega)$ and phase $\Phi(\omega)$, where $G(j\omega) = A(\omega) e^{j\Phi(\omega)}$.

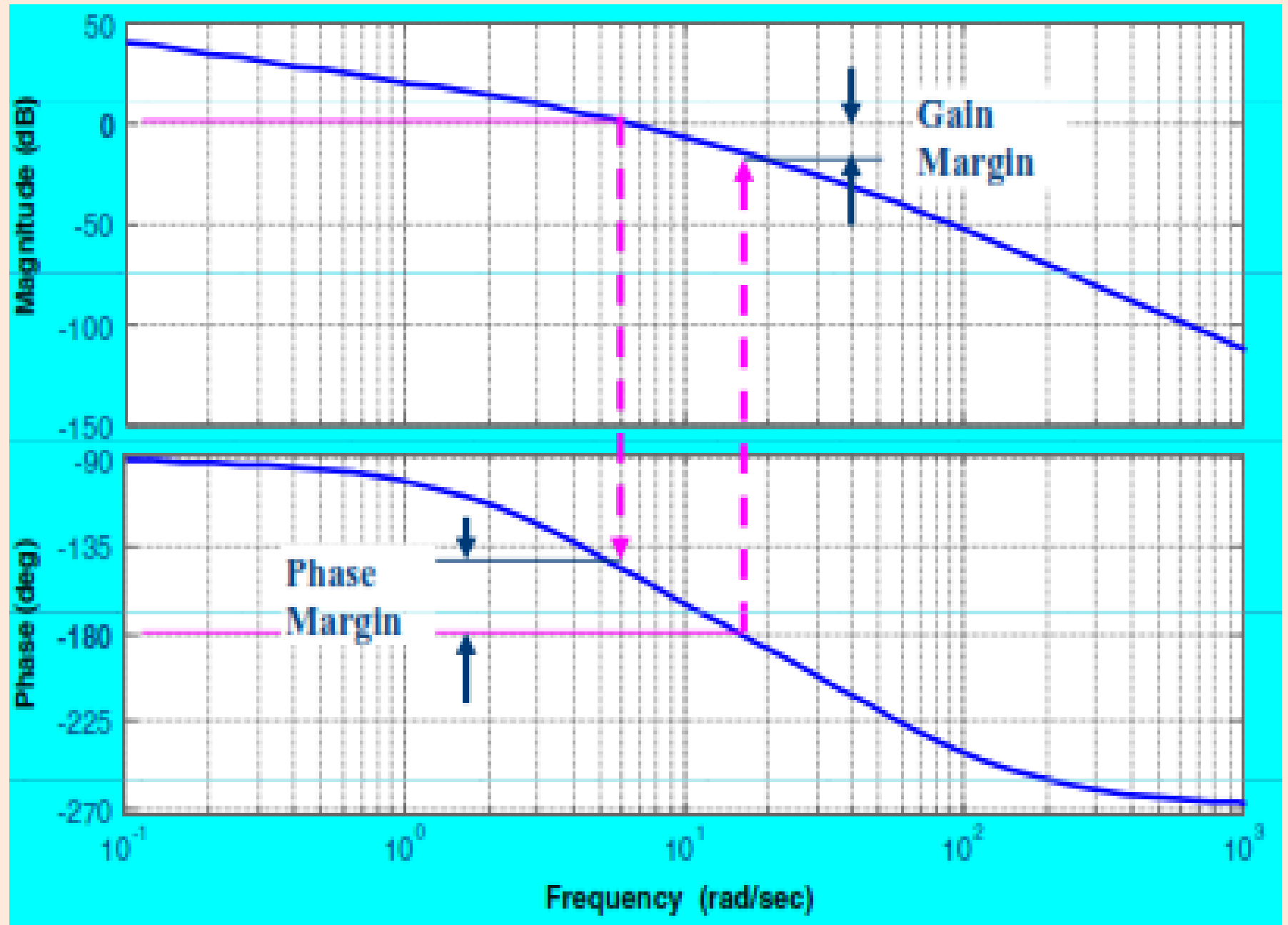
BODE PLOTS

The Bode plotting technique allows plotting frequency response in an efficient way and it is sufficient for control system design.

- 1) magnitude curve (log. Scale **dB**)
vs. frequency (log. Scale **radians/second**)
- 2) phase curve (linear scale **degree or radian**)
vs. frequency (log. scale)

This allows the plotting of a high-order $G(j\omega)$ by simple graphical addition of the separate terms.

BODE PLOTS



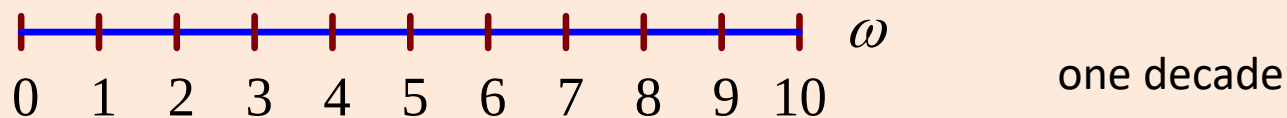
- A **Bode plot** is a (semilog) plot of the transfer function magnitude and phase angle as a function of frequency
- The gain magnitude is many times expressed in terms of decibels (dB)

$$\text{dB} = 20 \log_{10} A$$

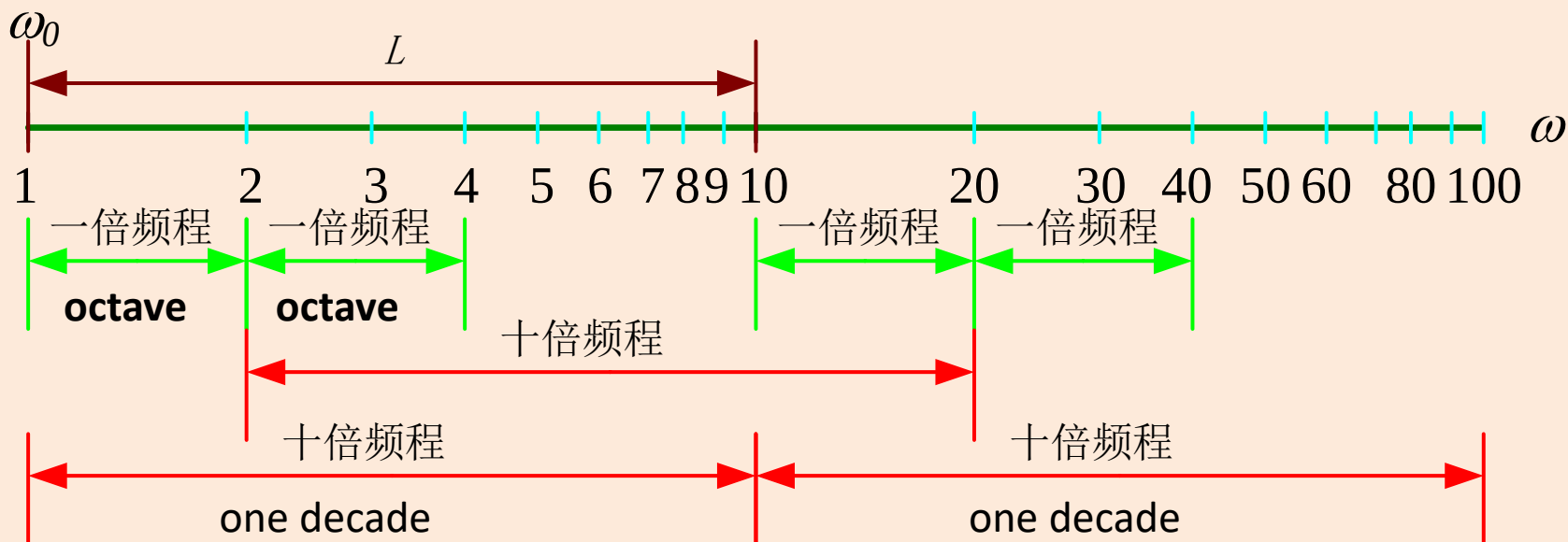
where A is the amplitude or gain

- a *decade* is defined as any 10-to-1 frequency range
 - an *octave* is any 2-to-1 frequency range
- $$20 \text{ dB/decade} = 6 \text{ dB/octave}$$

- Log scale and linear scale



(a) 线性分度



(b) 对数分度

Bode Plots

- Express the transfer function in standard form

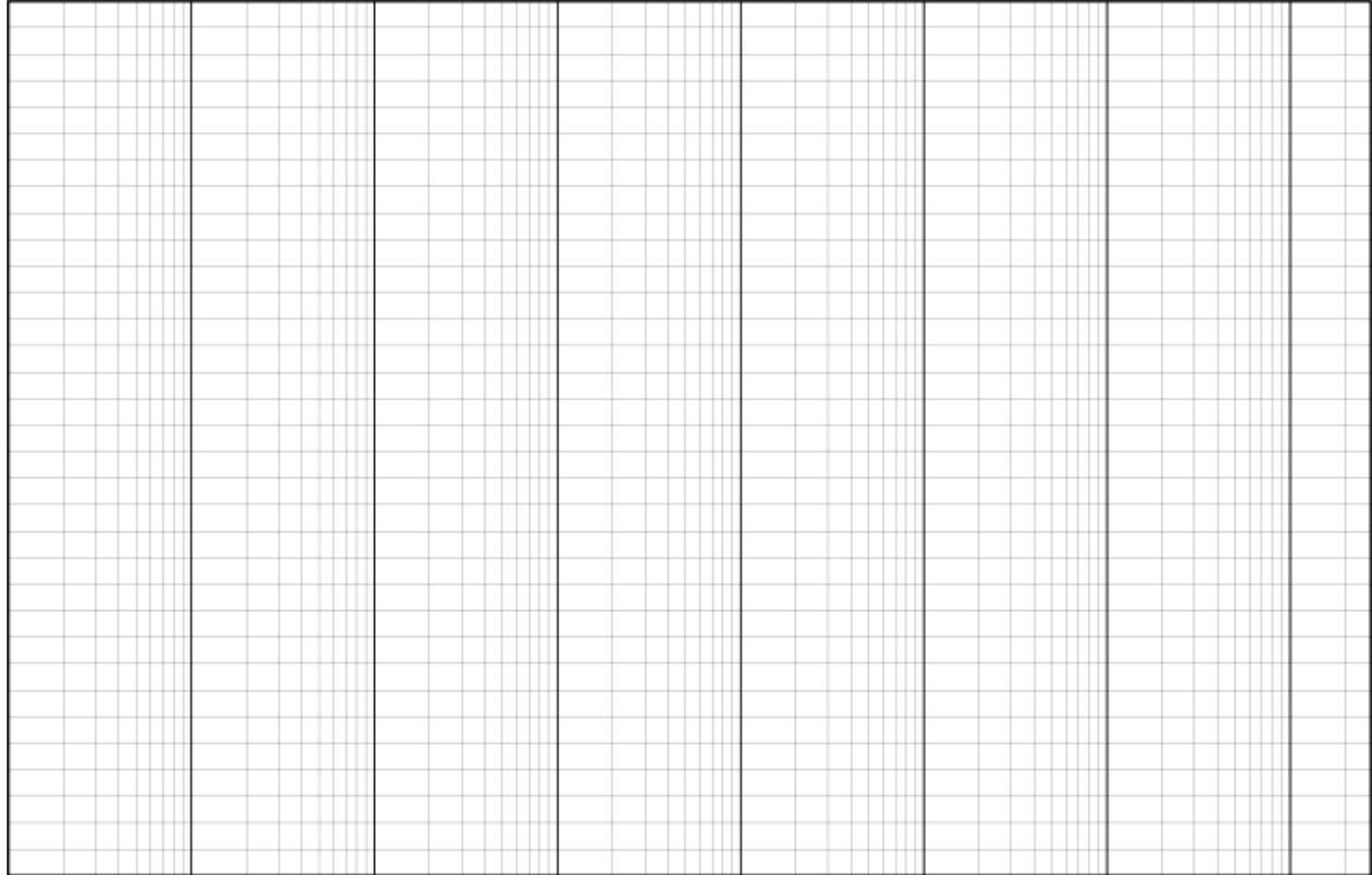
$$G(j\omega) = \frac{K(j\omega)^{\pm N} (1 + j\omega\tau_1) [1 + 2\zeta_2(j\omega\tau_2) + (j\omega\tau_2)^2] \cdots}{(1 + j\omega\tau_a) [1 + 2\zeta_b(j\omega\tau_b) + (j\omega\tau_b)^2] \cdots}$$

- There are four different factors:
 - Constant gain term, K
 - Poles or zeros at the origin, $(j\omega)^{\pm N}$
 - Poles or zeros of the form $(1 + j\omega\tau)$
 - Quadratic poles or zeros of the form $1 + 2\zeta(j\omega\tau) + (j\omega\tau)^2$

Bode Plots Methods

- هناك طريقتان لرسم البود بلوت :
- الطريقة الاولى: نأخذ دالة انتقال الدارة المفتوحة OPEN LOOP T.F=GH ونطبق الخطوات الاتية:
- نعوض عن كل S ب $j\omega$.
- نحسب $Lm=20\log GH$ ويقاس ب db .
- نحسب $\Phi = \text{زاوية } GH$ وتقاس ب degree .
- نعمل جدول بين ω, Lm, Φ حيث نعطي قيم ω ونحسب Lm, Φ
- الخطوة الاخيرة نرسم البود بلوت باستخدام ورقة بيانية تسمى
- Semi log graph paper الموضحة في الشريحة اللاحقة.

Semi log graph paper



- نأخذ مثال على الطريقة الاولى : For a unity negative feedback system having loop transfer function:- Plot on semi log paper the complete Bode plots.

$$G(s)H(s) = \frac{30(1+0.5s)}{s(0.1s+1)(0.2s+1)}$$

$$G(s)H(s) = \frac{30(1+0.5s)}{s(0.1s+1)(0.2s+1)} \dots\dots\dots(1)$$

Sub, $s=j\omega$ in eqn.(1) to get a frequency function

$$G(j\omega)H(j\omega) = \frac{30(1+0.5j\omega)}{j\omega(0.1j\omega+1)(0.2j\omega+1)} \dots\dots\dots(2)$$

Use $H(j\omega) = 1$ for unity feedback.

The log magnitude (LM) from eqn.(2) is :

$$LMGH(j\omega) = 20\log 30 + 20 \log|(1 + 0.5j\omega)| - 20\log|j\omega| \\ - 20\log|(0.1j\omega + 1)| - 20\log|(0.2j\omega + 1)|$$

$$LMGH(j\omega) = 20\log 30 + 20 \log^2\sqrt{(1)^2 + (0.5\omega)^2} - 20\log^2\sqrt{\omega^2} \\ - 20 \log^2\sqrt{(1)^2 + (0.1\omega)^2} - 20 \log^2\sqrt{(1)^2 + (0.2\omega)^2}$$

$$LMGH(j\omega) = 29.54 + 10 \log(1 + (0.25\omega^2)) - 20\log\omega - 10\log(1 + 0.01\omega^2) - 10\log(1 + 0.04\omega^2) \dots\dots\dots(3) \text{ db.}$$

The phase angle in degree from eqn.(2) is.

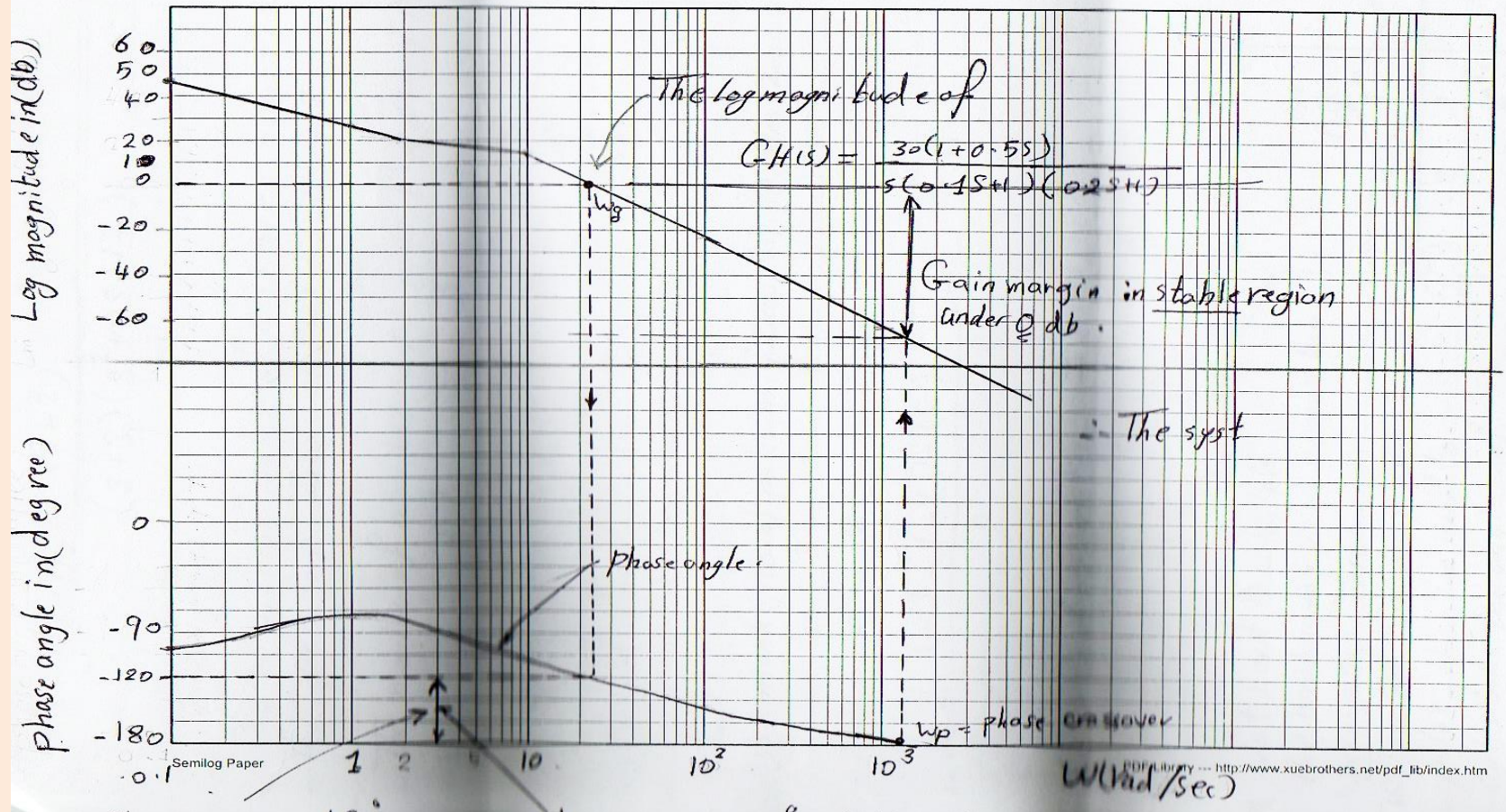
$$\phi(\omega) = \tan^{-1} 0.5\omega - 90 - \tan^{-1} 0.1\omega - \tan^{-1} 0.2\omega \dots\dots\dots(4) \text{ degree.}$$

to plot the Bode diagram using the general equation (3 & 4) with the various values of ω to make a table.

From table (1) : The complete Bode plot with checking the

ω (rad/sec)	L_m (db)	$\phi(\omega)$ degree
0.1	49.5	-99.169
1	28	-80.5
10	14	-119.5
100	-22.5	-172.9
1000	-62.5	-179.3

system stability
as shown in
semilog paper



Bode Plots Methods

- الطريقة الثانية: نأخذ كل حد من حدود الـ GH ونرسم له بود بلوت.
- نجمع كل الرسومات ونحصل على الرسم النهائي أو الكلي للبودبلوت .
- وكما موضح بالشريحة القادمة.

BODE PLOTS

Bode form of the transfer function

$$kG(s) = k_0 \frac{(b_1s + 1)(b_2s + 1) \dots \dots \dots}{s^l (a_1s + 1)(a_2s + 1) \dots \dots \dots}$$

For type 0 system and $s=j\omega$, k_0 is the gain at $\omega = 0$ and k_0 is called the dc gain of the system.

Classes of terms of transfer functions

(1). k_0

(3). $(j\frac{\omega}{\omega_c} + 1)^{\pm 1}$

(2). $(j\omega)^{\pm n}$

(4). $\left[\left(\frac{j\omega}{\omega_n} \right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \right]^{\pm 1}$

BODE PLOT

Gain Factor K.

- The gain factor multiplies the overall gain by a constant value for all frequencies.
- It has no effect on phase.

$$G(s) = K$$

$$G(j\omega) = K$$

$$M = 20\log|G(j\omega)|$$
$$= 20\log(K) \text{ [dB]}$$

$$\phi = 0$$



M : magnitude, φ : phase.

BODE PLOT

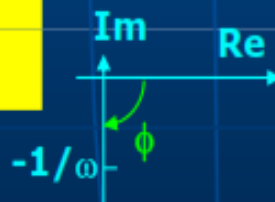
Integral Factor $1/j\omega$ – pole at the origin.

- **Magnitude is a straight line with a slope of -20 dB/decade becoming zero at $\omega=1$ [rad/s].**
- **Phase is constant at -90° at all frequencies.**

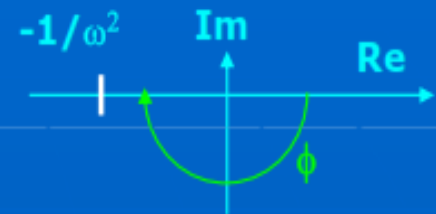
$$G(s) = \frac{1}{s}, \quad G(j\omega) = \frac{1}{j\omega} = -\frac{1}{\omega}j$$

$$M = 20\log|G(j\omega)|$$
$$= 20\log\left(\frac{1}{\omega}\right) = -20\log\omega$$

$$\phi = -90^\circ$$



BODE PLOT



Double pole at the origin.

- Simply double the slope of the magnitude and the phase, i.e., -40 dB/decade becoming zero at $\omega=1$ [rad/s] and -180° phase.

$$G(s) = \frac{1}{s^2}, \quad G(j\omega) = \frac{1}{(j\omega)^2} = -\frac{1}{\omega^2}$$

$$\begin{aligned} M &= 20\log|G(j\omega)| \\ &= 20\log\left(\frac{1}{\omega^2}\right) = -40\log\omega \\ \phi &= -180^\circ \end{aligned}$$



BODE PLOT

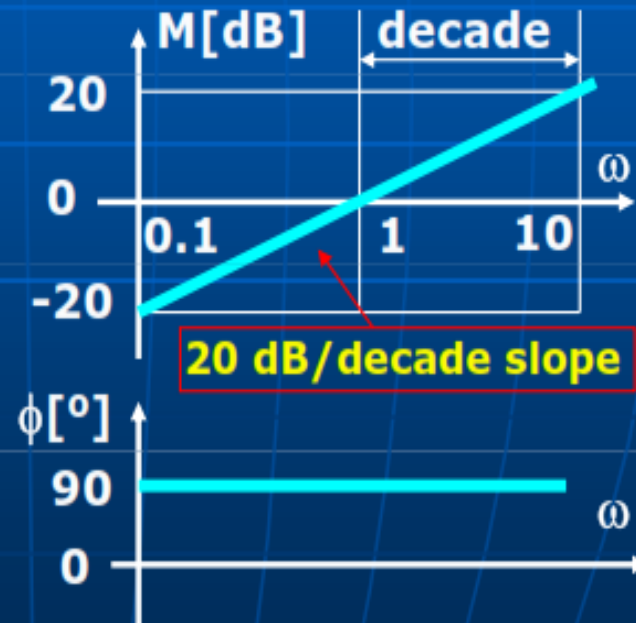
Derivative Factor $j\omega$ – zero at the origin.

- **Magnitude is a straight line with a slope of 20 dB/decade becoming zero at $\omega=1$ [rad/s].**
- **Phase is constant at 90° at all frequencies.**

$$G(s) = s, \quad G(j\omega) = \omega j$$

$$M = 20 \log |G(j\omega)| \\ = 20 \log(\omega)$$

$$\phi = 90^\circ$$



BODE PLOT

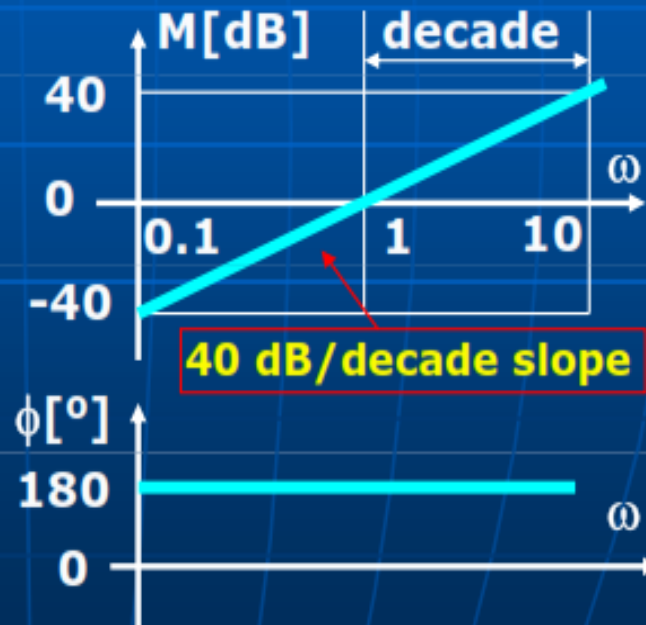
Double zero at the origin.

- Simply double the slope of the magnitude and the phase, i.e., 40 dB/decade becoming zero at $\omega=1$ [rad/s] and 180° phase.

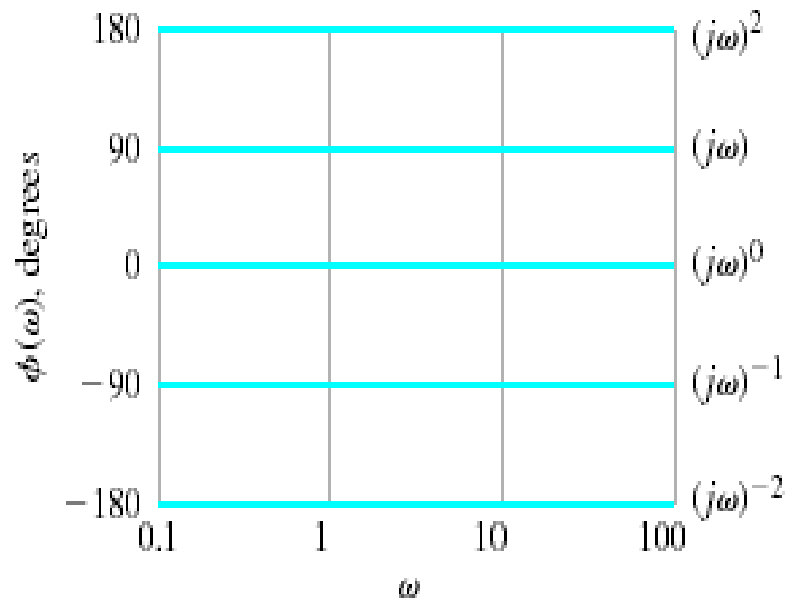
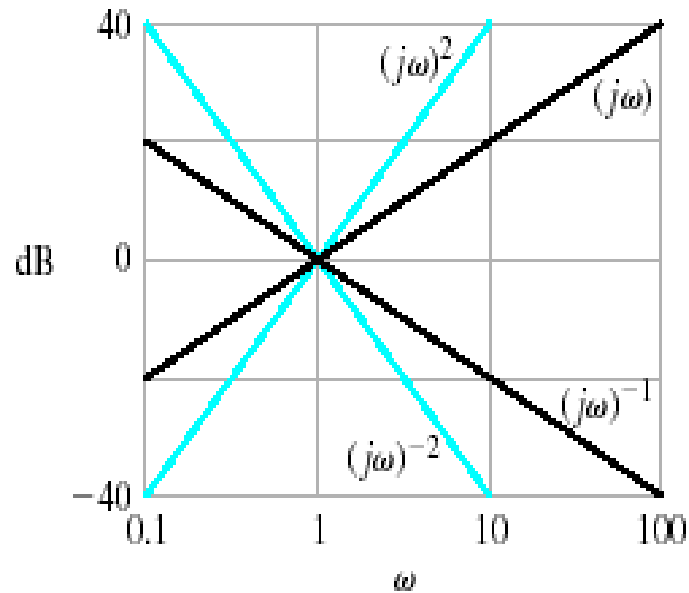
$$G(s) = s^2, \quad G(j\omega) = -\omega^2$$

$$M = 20\log|G(j\omega)| \\ = 40\log(\omega)$$

$$\phi = 180^\circ$$



Bode diagram for $(j\omega)^{\pm N}$.



BODE PLOT – First Order Factor

Simple lag (Real pole) $1/(1+j\omega T)$.

$$G(s) = \frac{1}{1 + Ts}$$

$$G(j\omega) = \frac{1}{1 + j\omega T} \frac{1 - j\omega T}{1 - j\omega T} = \frac{1}{1 + \omega^2 T^2} - \frac{\omega T}{1 + \omega^2 T^2} j$$

$$M = 20 \log |G(j\omega)| = 20 \log \left(\frac{1}{\sqrt{1 + \omega^2 T^2}} \right)$$

$$M = -20 \log \sqrt{1 + \omega^2 T^2} \text{ [dB]}$$

$$\phi = \tan^{-1}(-\omega T) = -\tan^{-1} \omega T$$

BODE PLOT – First Order Factor

Simple lag (Real Pole) $1/(1+j\omega T)$.

$$M = -20 \log \sqrt{1 + \omega^2 T^2} \text{ [dB]}$$



For $\omega \ll \frac{1}{T}$

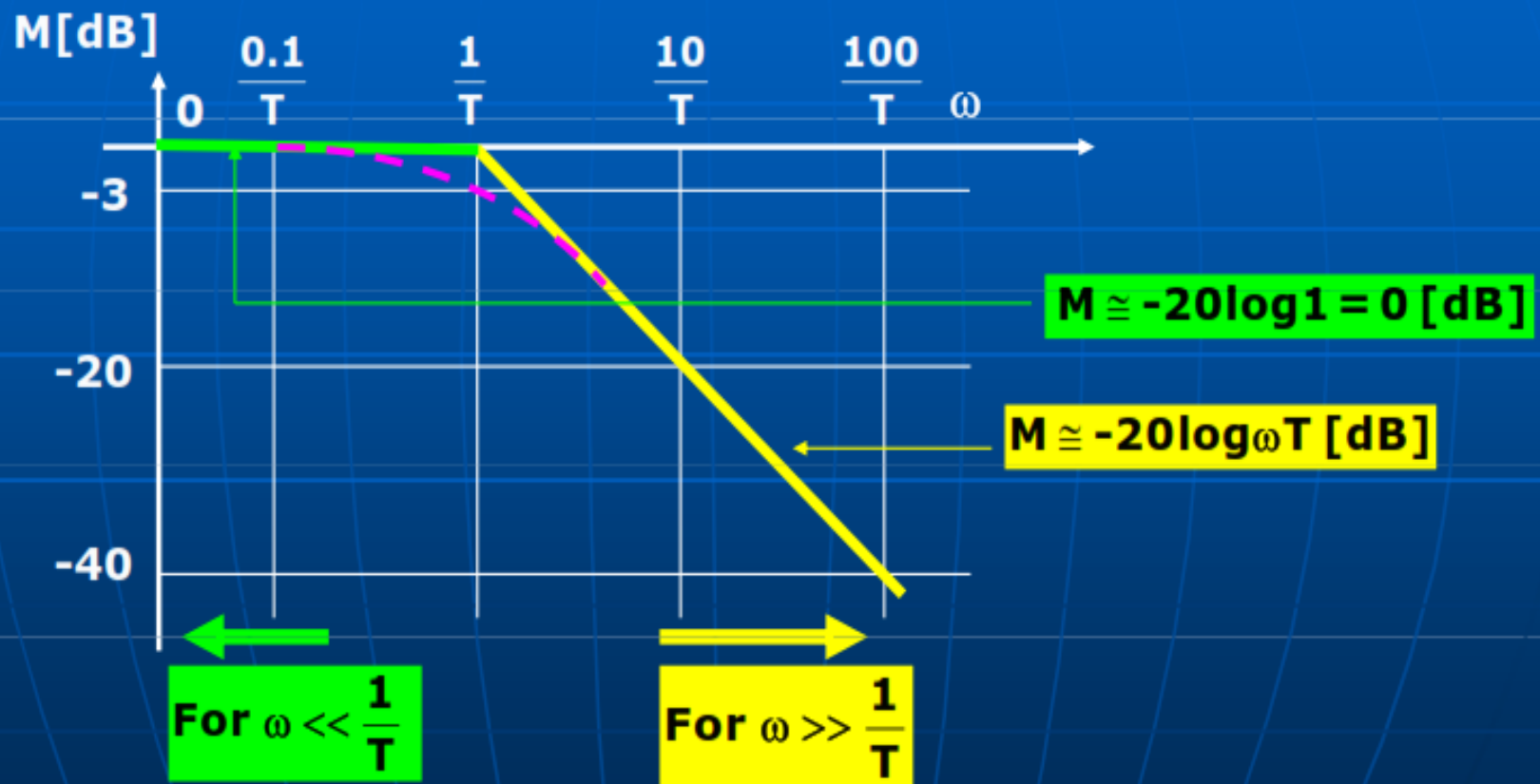
$$M \cong -20 \log 1 = 0 \text{ [dB]}$$

For $\omega \gg \frac{1}{T}$

$$M \cong -20 \log \omega T \text{ [dB]}$$

BODE PLOT – First Order Factor

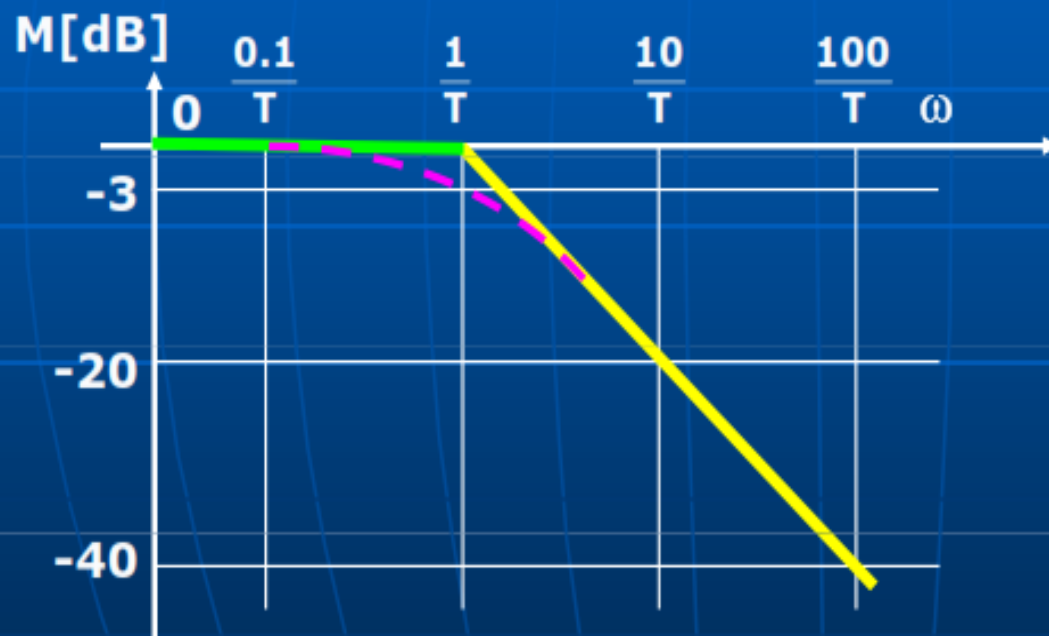
It is clear that the actual magnitude curve can be approximated by two straight lines.



BODE PLOT – First Order Factor

$\omega_c = 1/T$ is called the **corner (break) frequency**.
Maximum error between the linear approximation and the exact value will be at the corner frequency.

$$M = -20 \log \sqrt{1 + \omega^2 T^2} \text{ [dB]}$$

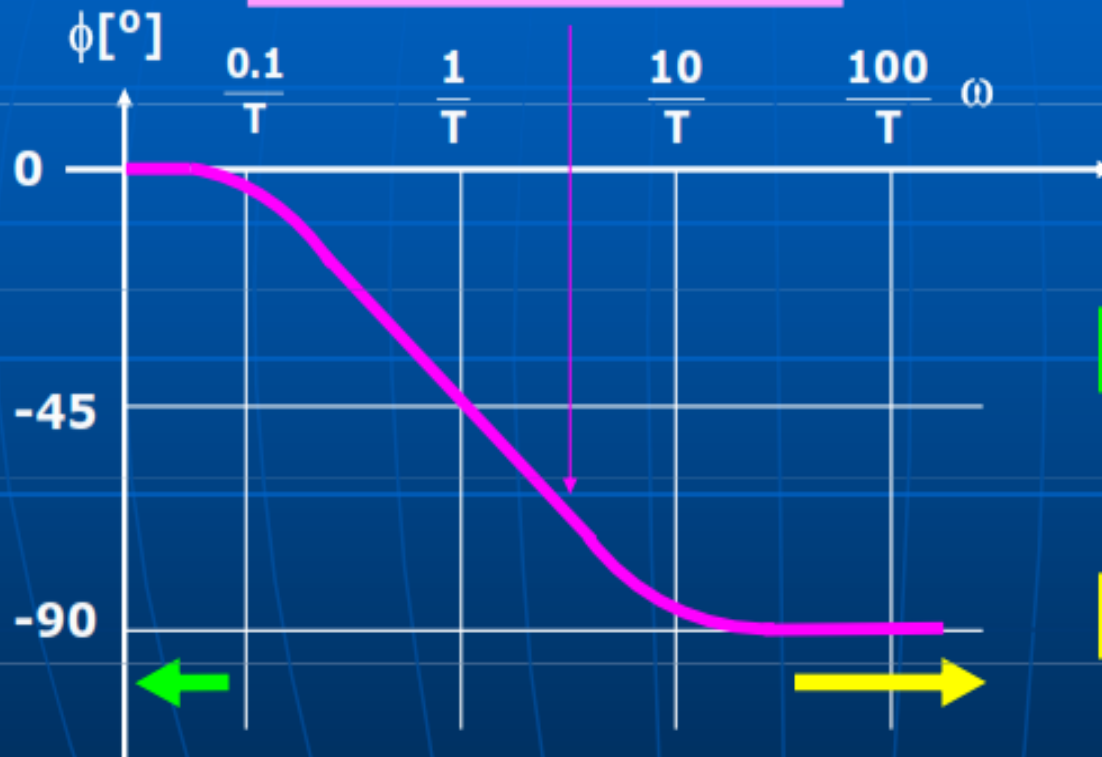


$$M\left(\omega = \frac{1}{T}\right) = -20 \log \sqrt{2} \\ \approx -3[\text{dB}]$$

BODE PLOT – First Order Factor

Simple lag $1/(1+j\omega T)$.

$$\phi = \tan^{-1}(-\omega T) = -\tan^{-1}\omega T$$



For $\omega \ll \frac{0.1}{T}$

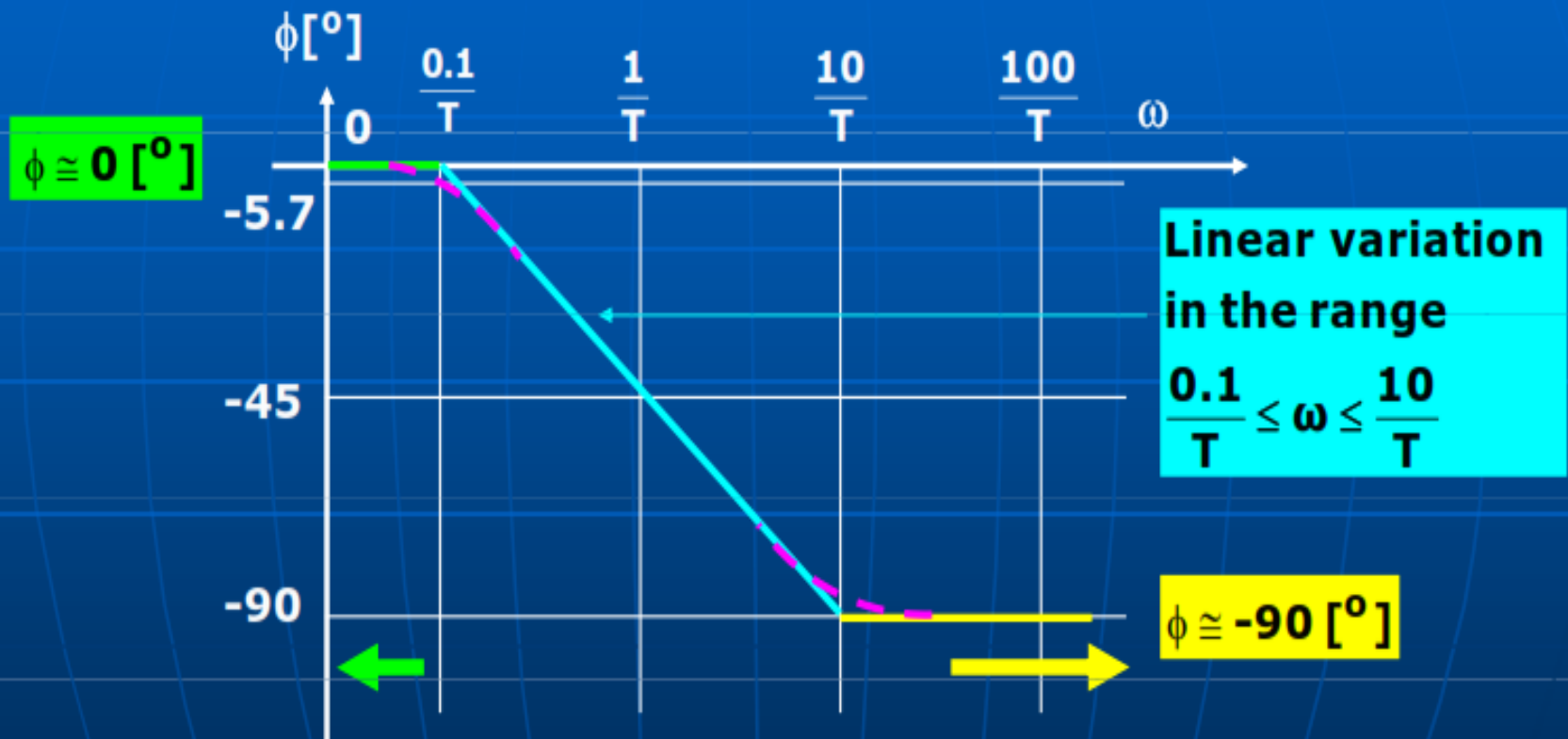
$$\phi \approx 0 [^\circ]$$

For $\omega \gg \frac{10}{T}$

$$\phi \approx -90 [^\circ]$$

BODE PLOT – First Order Factor

It is clear that the actual phase curve can be approximated by three straight lines.



In this case corner frequencies are : $0.1/T$ and $10/T$

BODE PLOT – First Order Factor

Simple lead (Real zero) $1+j\omega T$.

$$G(s) = 1 + Ts$$

$$G(j\omega) = 1 + \omega Tj$$

$$M = 20\log|G(j\omega)| = 20\log\left(\sqrt{1 + \omega^2 T^2}\right)$$

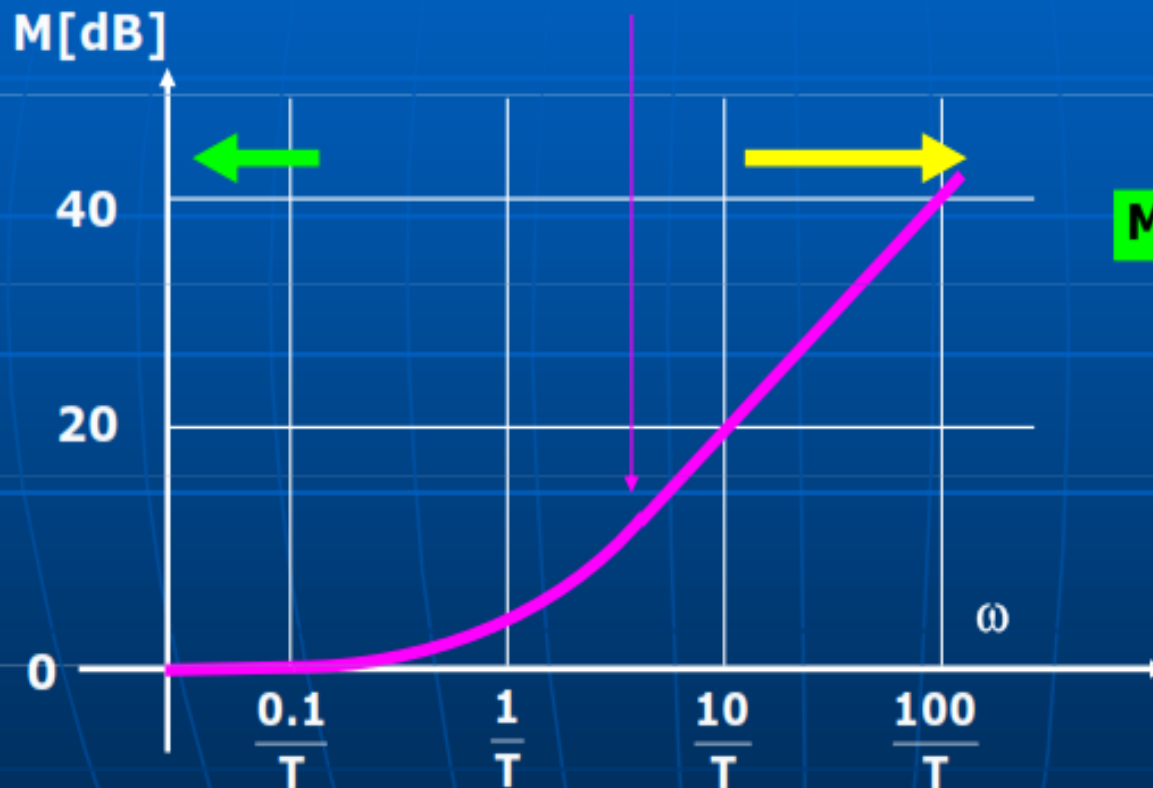
$$M = 20\log\sqrt{1 + \omega^2 T^2} \text{ [dB]}$$

$$\phi = \tan^{-1}(\omega T) = \tan^{-1}\omega T$$

BODE PLOT – First Order Factor

Simple lead (Real zero) $1+j\omega T$.

$$M = 20\log\sqrt{1 + \omega^2 T^2} \text{ [dB]}$$



For $\omega \ll \frac{1}{T}$

$$M \approx 20\log 1 = 0 \text{ [dB]}$$

For $\omega \gg \frac{1}{T}$

$$M \approx 20\log \omega T \text{ [dB]}$$

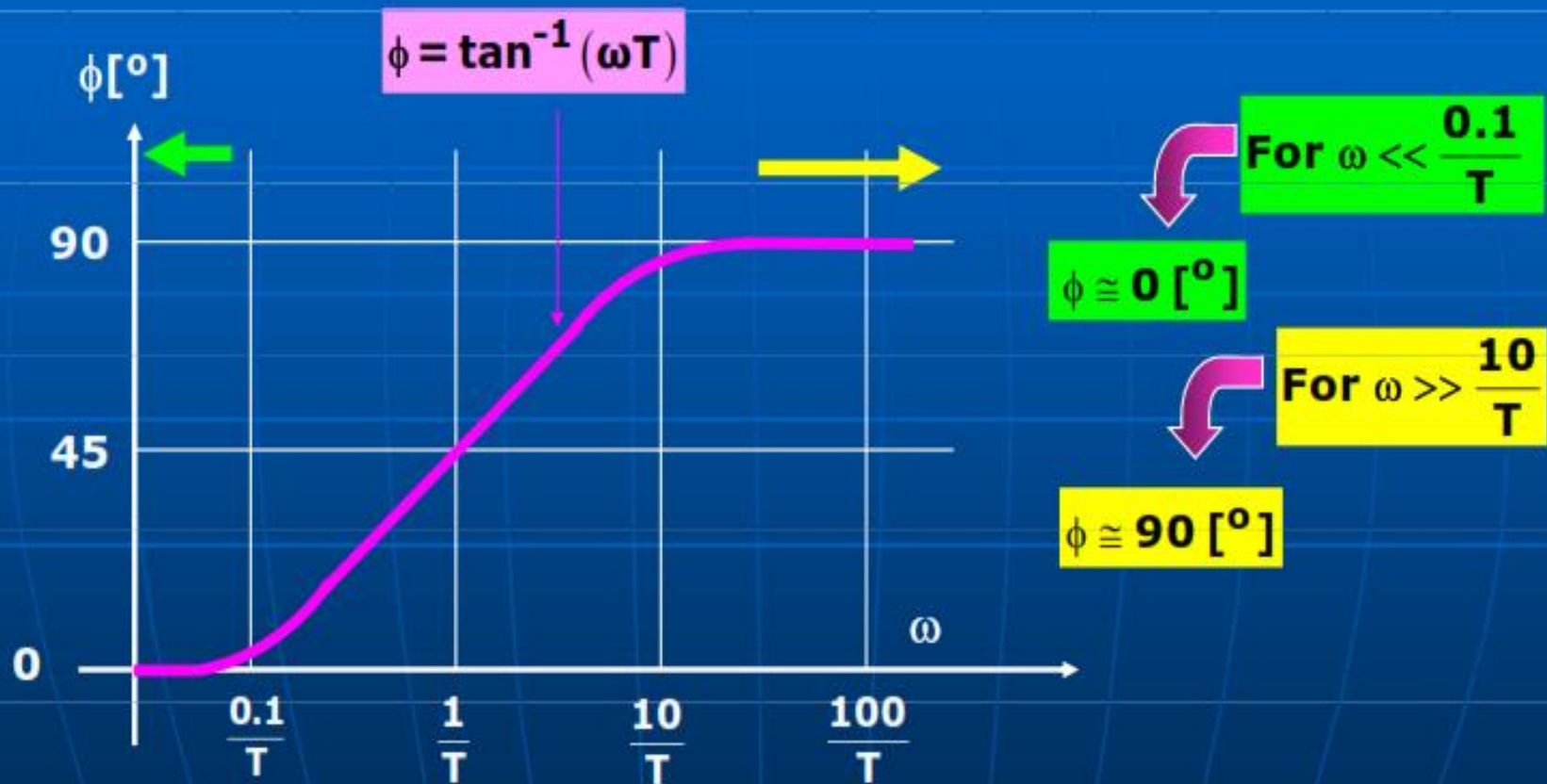
BODE PLOT – First Order Factor

It is clear that the actual magnitude curve can be approximated by two straight lines.



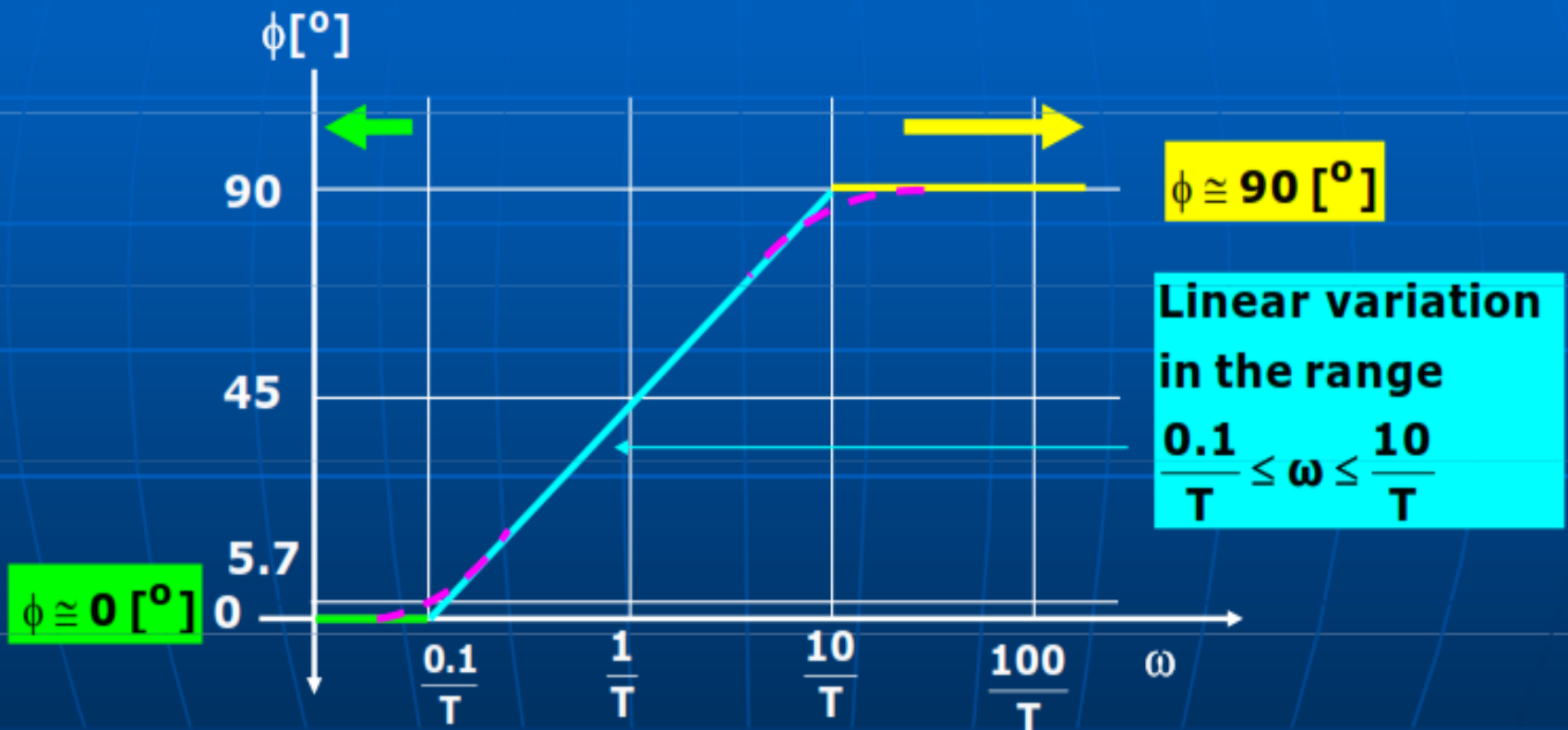
BODE PLOT – First Order Factor

Simple lead $1+j\omega T$.



BODE PLOT – First Order Factor

It is clear that the actual phase curve can be approximated by three straight lines.



Rules for Making Bode Plots

Term	Magnitude	Phase
Constant: K	$20 \cdot \log_{10}(K)$	K>0: 0° K<0: $\pm 180^\circ$
Real Pole: $\frac{1}{\frac{s}{\omega_0} + 1}$	- Low freq. asymptote at 0 dB - High freq. asymptote at -20 dB/dec - Connect asymptotic lines at ω_0.	- Low freq. asymptote at 0°. - High freq. asymptote at -90°. - Connect with straight line from $0.1 \cdot \omega_0$ to $10 \cdot \omega_0$.
Real Zero: $\frac{s}{\omega_0} + 1$	- Low freq. asymptote at 0 dB - High freq. asymptote at +20 dB/dec. - Connect asymptotic lines at ω_0.	- Low freq. asymptote at 0°. - High freq. asymptote at $+90^\circ$. - Connect with line from $0.1 \cdot \omega_0$ to $10 \cdot \omega_0$.
Pole at Origin: $\frac{1}{s}$	-20 dB/dec; through 0 dB at $\omega=1$.	-90° for all ω.
Zero at Origin: s	+20 dB/dec; through 0 dB at $\omega=1$.	$+90^\circ$ for all ω.
Underdamped Poles: $\frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1}$	- Low freq. asymptote at 0 dB. - High freq. asymptote at -40 dB/dec. - Connect asymptotic lines at ω_0. - Draw peak[†] at freq. ω_0, with amplitude $H(j\omega_0) = -20 \cdot \log_{10}(2\zeta)$	- Low freq. asymptote at 0°. - High freq. asymptote at -180°. - Connect with straight line from $\omega = \omega_0 \cdot 10^{-\zeta}$ to $\omega_0 \cdot 10^\zeta$
Underdamped Zeros: $\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1$	- Low freq. asymptote at 0 dB. - High freq. asymptote at +40 dB/dec. - Connect asymptotic lines at ω_0. - Draw dip[†] at freq. ω_0, with amplitude $H(j\omega_0) = +20 \cdot \log_{10}(2\zeta)$	- Low freq. asymptote at 0°. - High freq. asymptote at $+180^\circ$. - Connect with straight line from $\omega = \omega_0 \cdot 10^{-\zeta}$ to $\omega_0 \cdot 10^\zeta$

BODE PLOT – Quadratic Factors

As overdamped systems can be replaced by two first order factors, only underdamped systems are of interest here.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

A set of two complex conjugate poles.

$$G(j\omega) = \frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\xi\left(j\frac{\omega}{\omega_n}\right) + 1}$$

$$M = 20\log|G(j\omega)| = -20\log\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2} \text{ [dB]}$$

BODE PLOT – Quadratic Factors

$$M = 20\log|G(j\omega)| = -20\log\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2} \text{ [dB]}$$

Low frequency asymptote, $\omega \ll \omega_n$:

$$M \cong -20\log(1) = 0 \text{ [dB]}$$

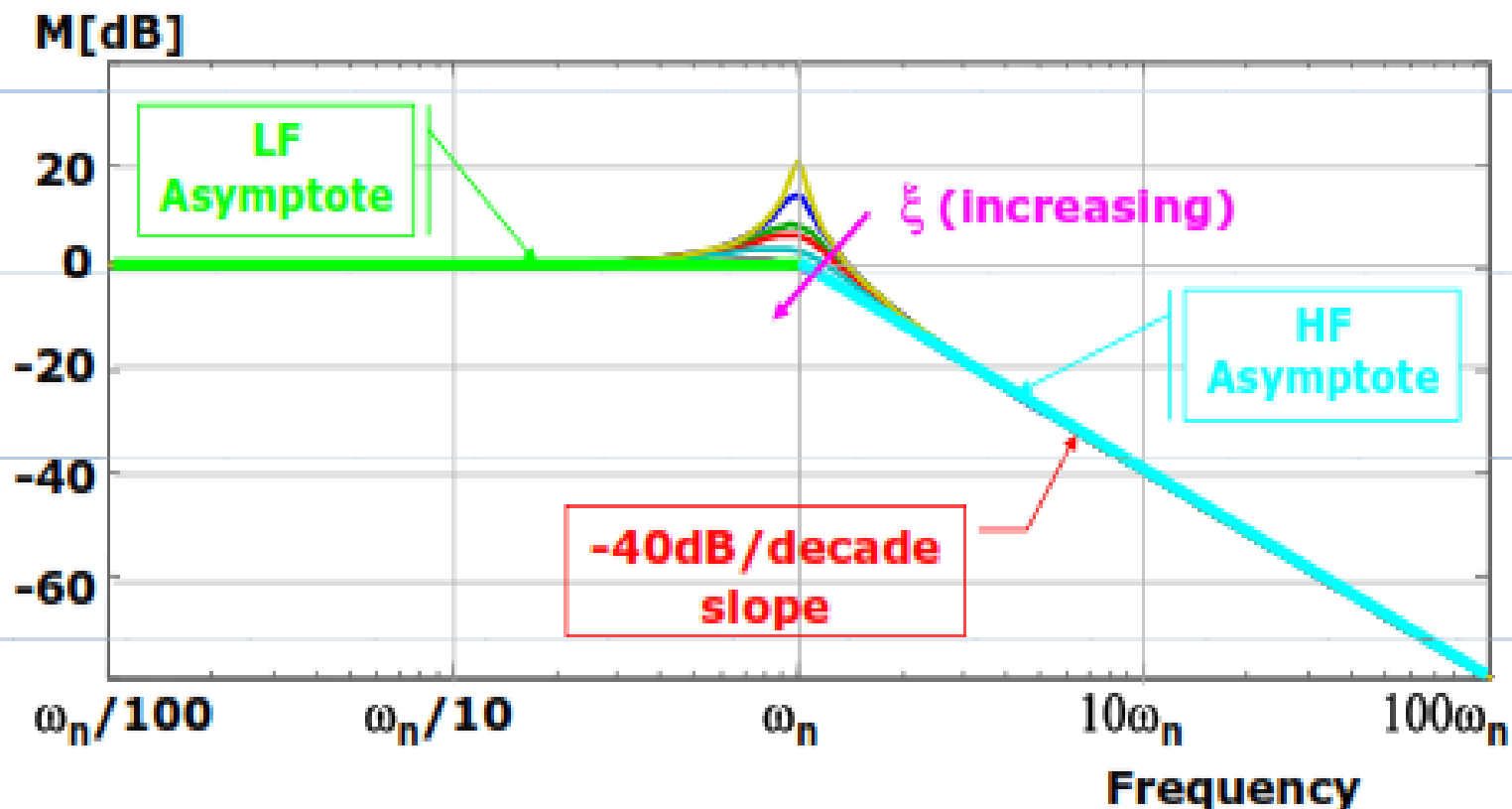
High frequency asymptote, $\omega \gg \omega_n$:

$$M \cong -20\log\left(\frac{\omega}{\omega_n}\right)^2 = -40\log\left(\frac{\omega}{\omega_n}\right) \text{ [dB]}$$

Low and high frequency asymptotes intersect at $\omega = \omega_n$, i.e. corner frequency is ω_n .

BODE PLOT – Quadratic Factors

Therefore the actual magnitude curve can be approximated by two straight lines.



BODE PLOT – Quadratic Factors

$$\phi = \angle G(j\omega) = -\tan^{-1} \frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

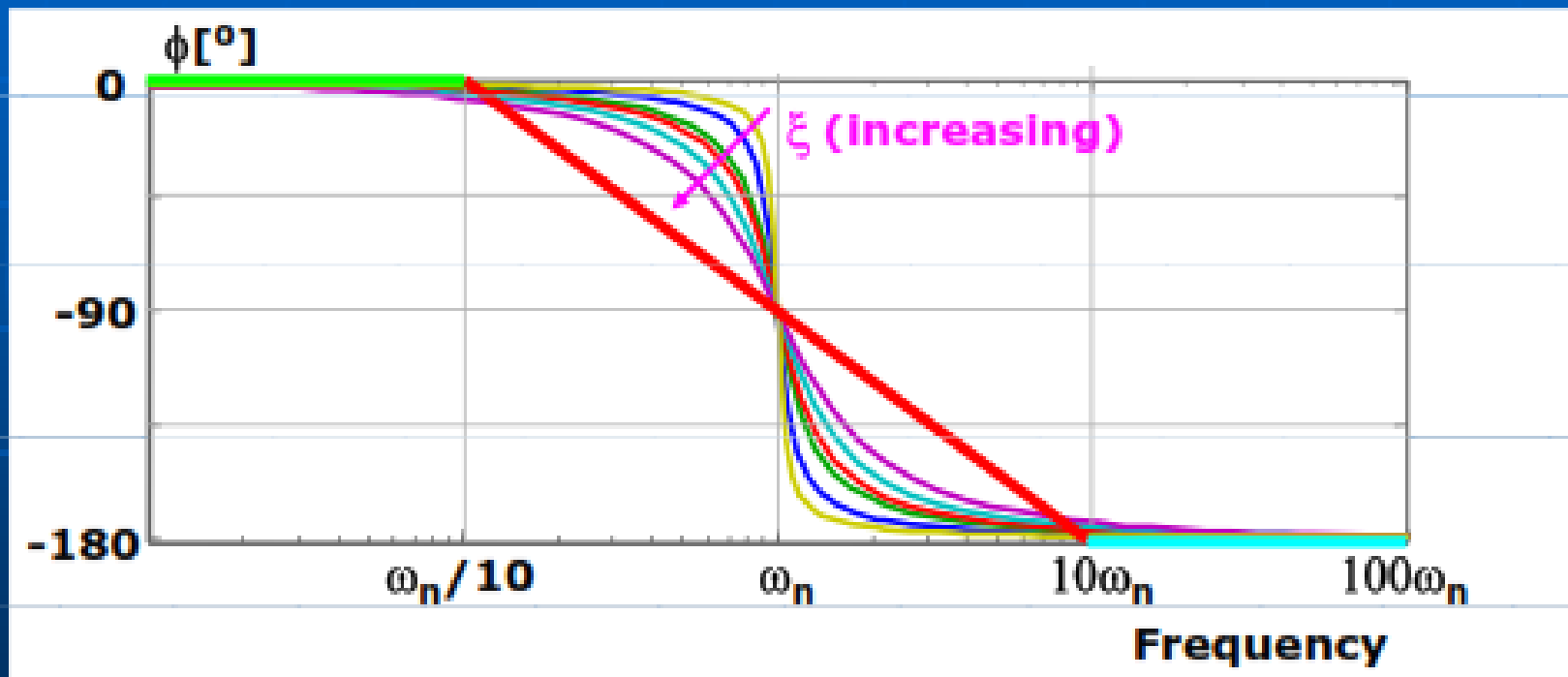
At low frequencies, $\omega \rightarrow 0$: $\phi \cong 0 [^\circ]$

At $\omega = \omega_n$: $\phi \cong -90 [^\circ]$

At high frequencies, $\omega \rightarrow \infty$: $\phi \cong -180 [^\circ]$

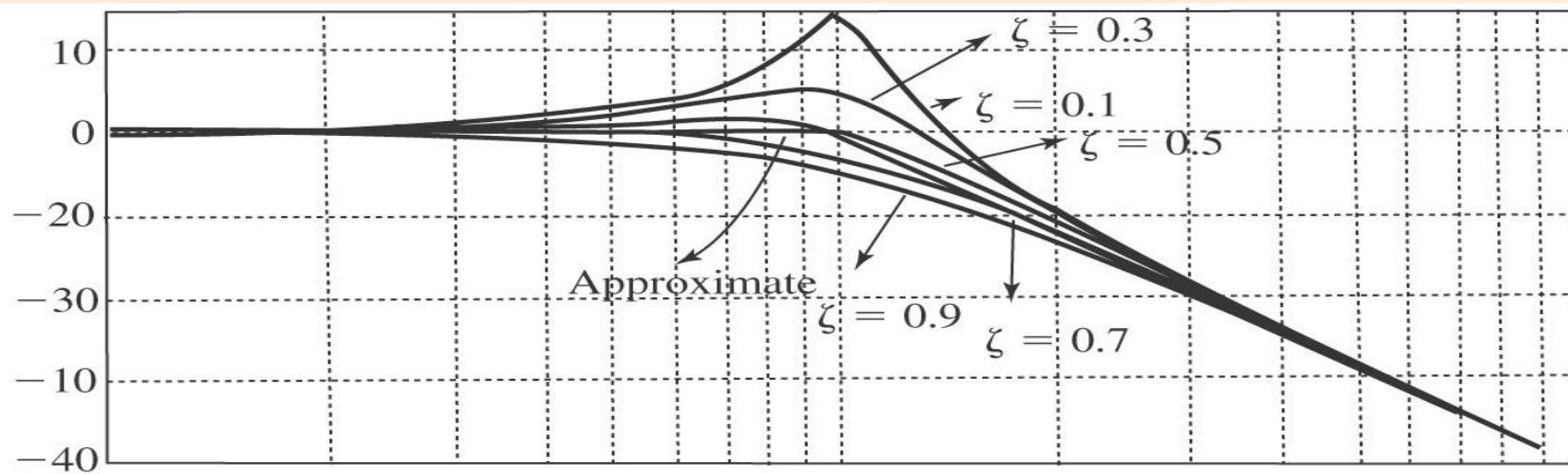
BODE PLOT – Quadratic Factors

Thus, the actual phase curve can be approximated by three straight lines.



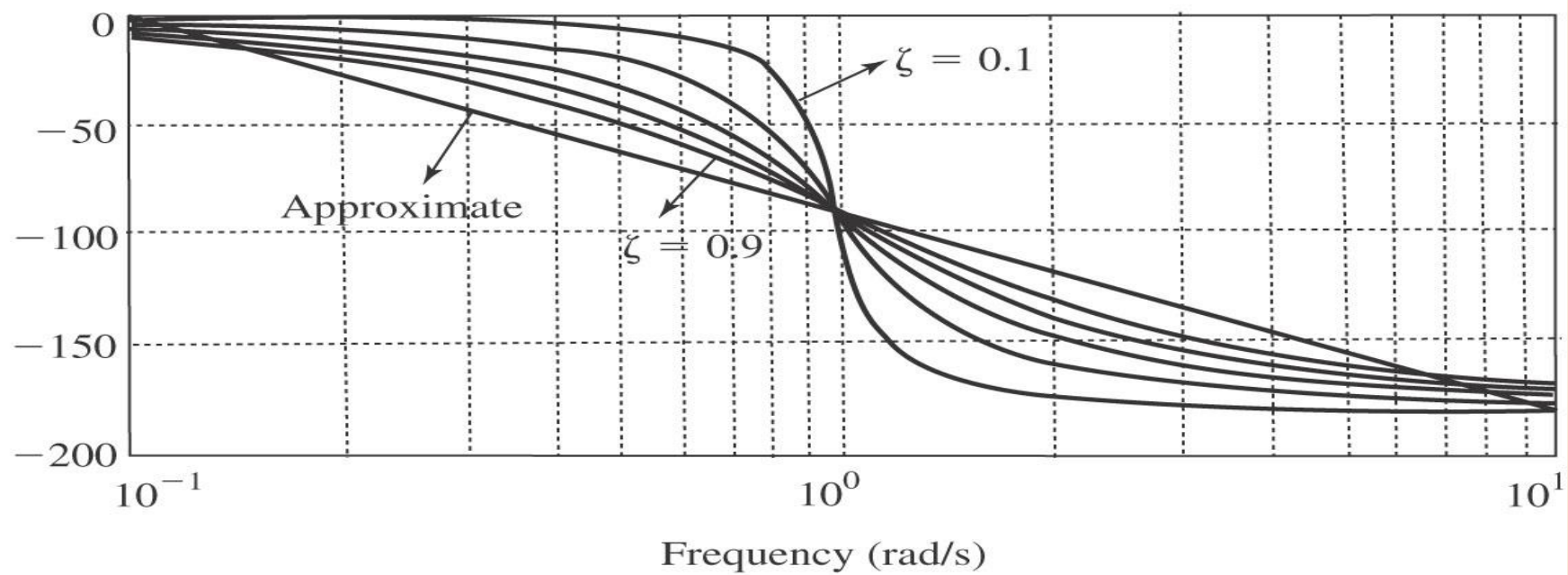
Corner frequencies are : $\omega_n/10$ and $10\omega_n$.

Magnitude dB



(a)

Phase



(b)

Rules for Making Bode Plots

Term	Magnitude	Phase
Constant: K	$20 \cdot \log_{10}(K)$	K>0: 0° K<0: $\pm 180^\circ$
Real Pole: $\frac{1}{\frac{s}{\omega_0} + 1}$	- Low freq. asymptote at 0 dB - High freq. asymptote at -20 dB/dec - Connect asymptotic lines at ω_0.	- Low freq. asymptote at 0°. - High freq. asymptote at -90°. - Connect with straight line from $0.1 \cdot \omega_0$ to $10 \cdot \omega_0$.
Real Zero: $\frac{s}{\omega_0} + 1$	- Low freq. asymptote at 0 dB - High freq. asymptote at +20 dB/dec. - Connect asymptotic lines at ω_0.	- Low freq. asymptote at 0°. - High freq. asymptote at $+90^\circ$. - Connect with line from $0.1 \cdot \omega_0$ to $10 \cdot \omega_0$.
Pole at Origin: $\frac{1}{s}$	-20 dB/dec; through 0 dB at $\omega=1$.	-90° for all ω.
Zero at Origin: s	+20 dB/dec; through 0 dB at $\omega=1$.	$+90^\circ$ for all ω.
Underdamped Poles: $\frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1}$	- Low freq. asymptote at 0 dB. - High freq. asymptote at -40 dB/dec. - Connect asymptotic lines at ω_0. - Draw peak[†] at freq. ω_0, with amplitude $H(j\omega_0) = -20 \cdot \log_{10}(2\zeta)$	- Low freq. asymptote at 0°. - High freq. asymptote at -180°. - Connect with straight line from $\omega = \omega_0 \cdot 10^{-5}$ to $\omega_0 \cdot 10^5$
Underdamped Zeros: $\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1$	- Low freq. asymptote at 0 dB. - High freq. asymptote at +40 dB/dec. - Connect asymptotic lines at ω_0. - Draw dip[†] at freq. ω_0, with amplitude $H(j\omega_0) = +20 \cdot \log_{10}(2\zeta)$	- Low freq. asymptote at 0°. - High freq. asymptote at $+180^\circ$. - Connect with straight line from $\omega = \omega_0 \cdot 10^{-5}$ to $\omega_0 \cdot 10^5$

BODE PLOT – Example 1a

- Sketch the Bode plots for the given open loop transfer function of a control system.

$$T(s) = \frac{100000(1+s)}{s(s+10)(0.1s^2+14s+1000)}$$

- First convert to standard form.

$$T(j\omega) = \frac{100000(1+j\omega)}{(j\omega)(10)(1+0.1\omega j)(1000) \left[\left(j\frac{\omega}{100} \right)^2 + 1.4 \left(j\frac{\omega}{100} \right) + 1 \right]}$$

$$T(j\omega) = \frac{10(1+j\omega)}{(j\omega)(1+0.1\omega j) \left[\left(j\frac{\omega}{100} \right)^2 + 1.4 \left(j\frac{\omega}{100} \right) + 1 \right]}$$

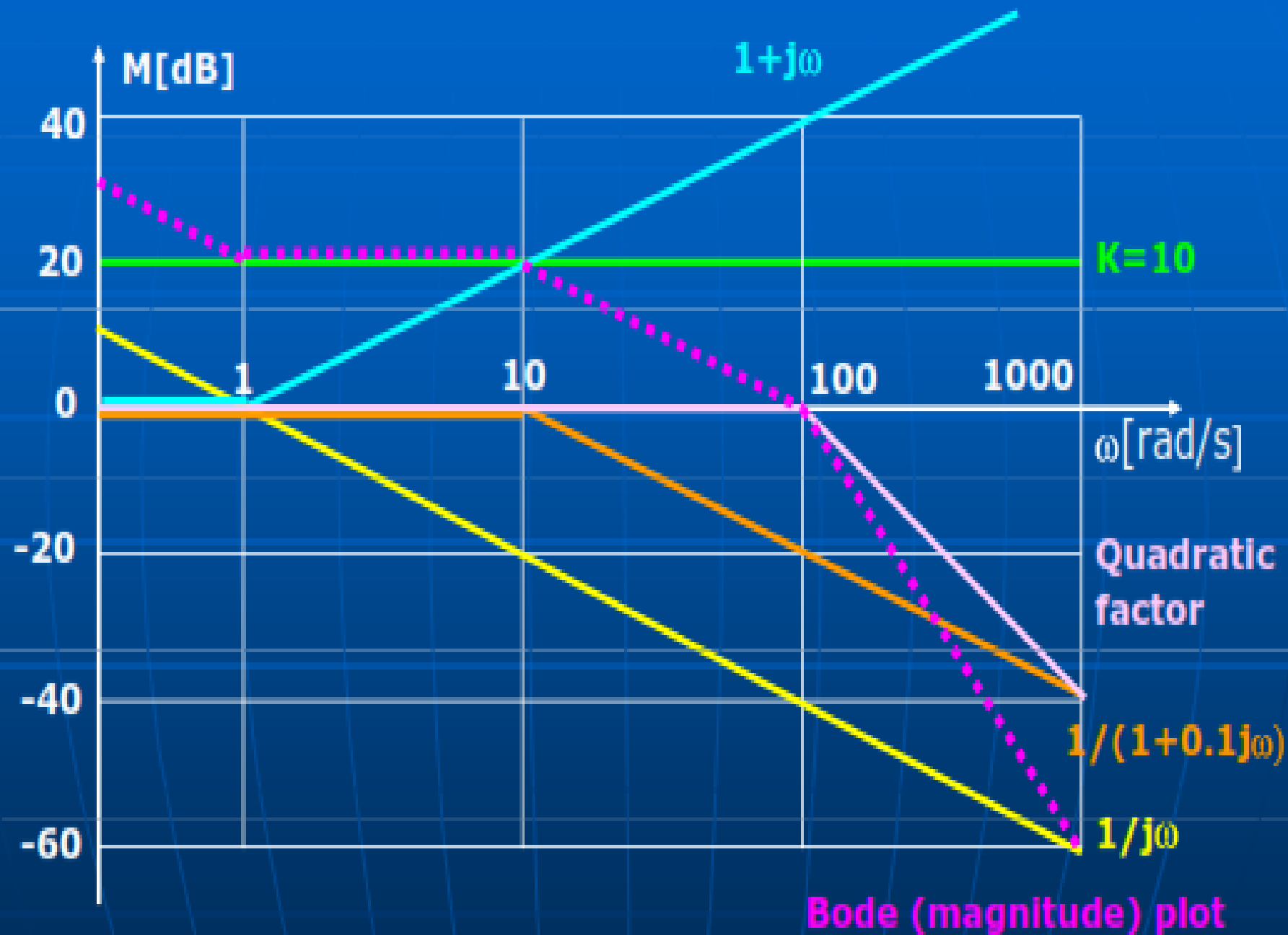
$$T(j\omega) = \frac{10(1+j\omega)}{(j\omega)(1+0.1\omega j) \left[\left(j \frac{\omega}{100} \right)^2 + 1.4 \left(j \frac{\omega}{100} \right) + 1 \right]}$$

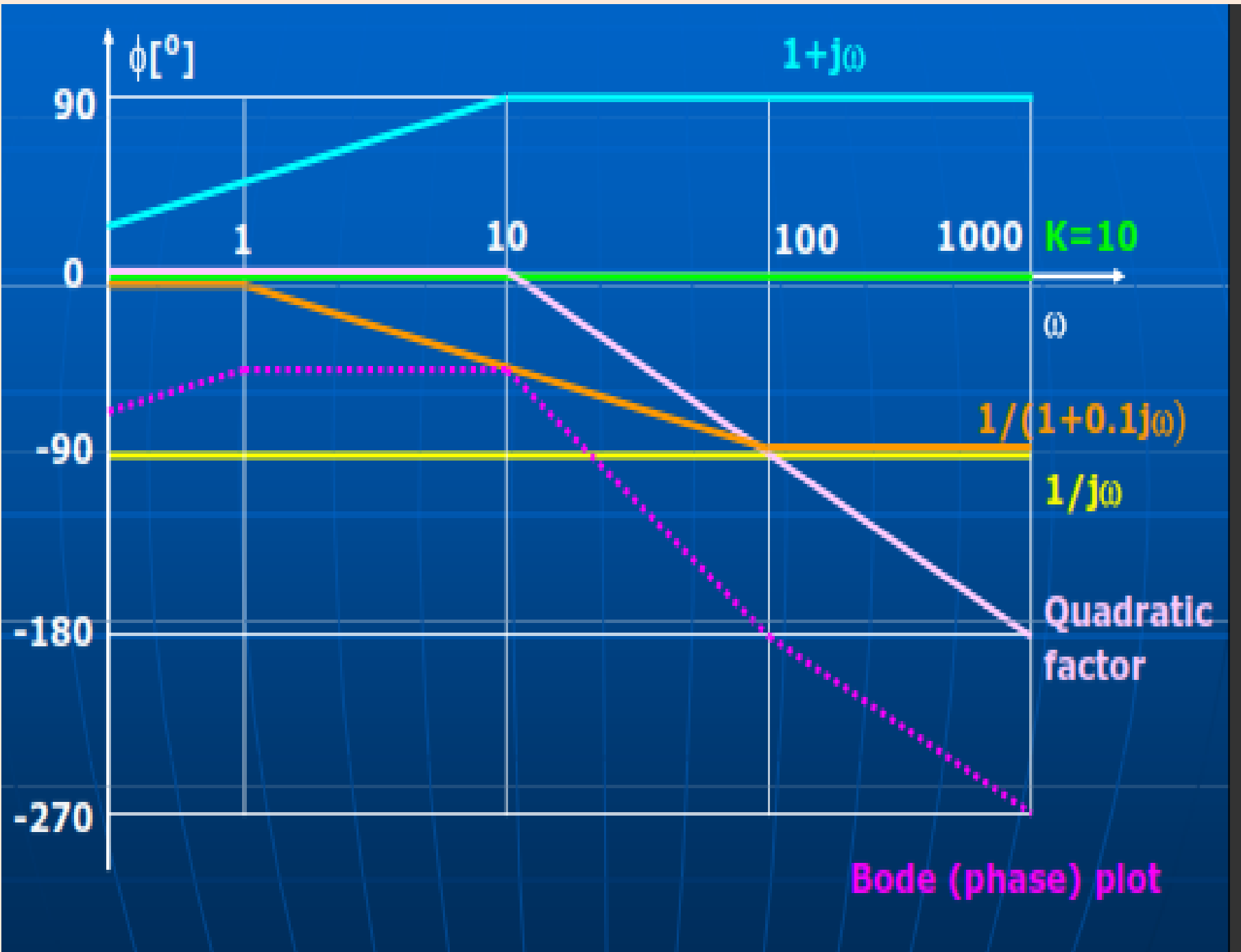
■ Identify the basic factors and corner frequencies :

- **Constant gain K** : $K=10$, $20\log 10=20$ [dB]
- **First order factor (simple lead – real zero)** :
 $T=1$ ($\omega_{c1}=1/T=1$) - for magnitude plot
- **Integral factor** : $1/j\omega$
- **First order factor (simple lag – real pole)** :
 $T=0.1$ ($\omega_{c1}=1/T=10$) - for magnitude plot
- **Quadratic factor (complex conjugate poles)** :
 $\omega_n=\omega_{c1}=100$, $\xi=0.7$ - for magnitude plot

$$T(j\omega) = \frac{10(1+j\omega)}{(j\omega)(1+0.1j\omega) \left[\left(j\frac{\omega}{100} \right)^2 + 1.4 \left(j\frac{\omega}{100} \right) + 1 \right]}$$

- Identify the basic factors and corner frequencies :
 - **Constant gain K** : $K=10$, $20\log 10=20$ [dB]
 - **First order factor (simple lead – real zero)** : $T=1$
 $(\omega_{c2}=0.1/T=0.1, \omega_{c3}=10/T=10)$ – for phase plot
 - **Integral factor** : $1/j\omega$
 - **First order factor (simple lag – real pole)** : $T=0.1$
 $(\omega_{c2}=0.1/T=1, \omega_{c3}=10/T=100)$ – for phase plot
 - **Quadratic factor (complex conjugate poles)** : $\omega_n=100$
 $(\omega_{c2}=\omega_n/10=10, \omega_{c3}=10\omega_n=1000)$ – for phase plot

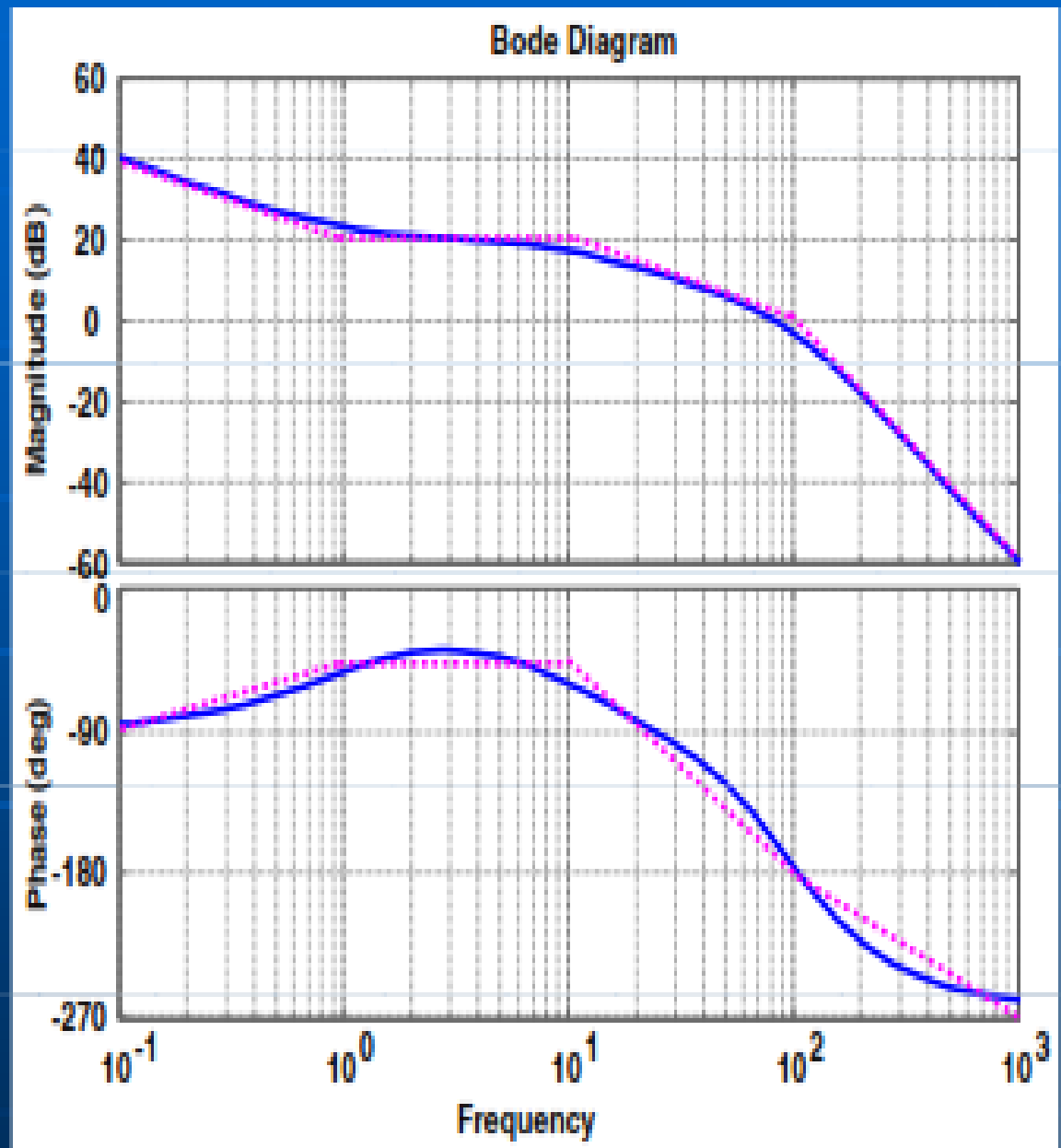




- Matlab plot:
full blue lines
(just 4 lines
to plot !)

```
num=[100000 100000]  
den=[0.1 15 1140 10000 0]  
bode(num,den)  
grid
```

Approximate
plots: dashed
lines



Relative stability

A transfer function is called **minimum phase** when all the poles and zeros are LHP and **non-minimum-phase** when there are RHP poles or zeros.

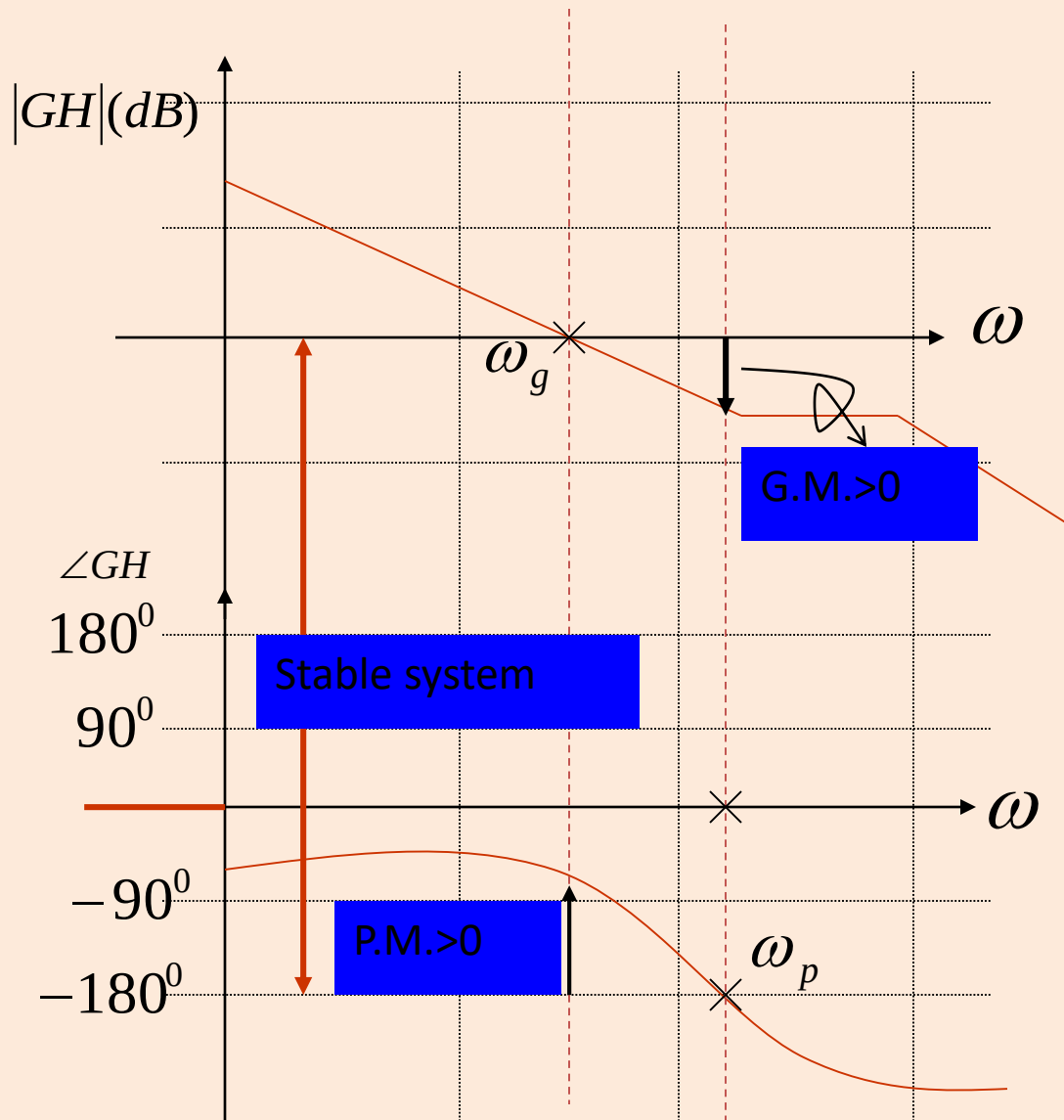
Minimum phase system



Stable

The **gain margin (GM)** is the distance on the bode magnitude plot from the amplitude at the phase crossover frequency up to the 0 dB point. **$GM = -(\text{dB of } GH \text{ measured at the phase crossover frequency})$**

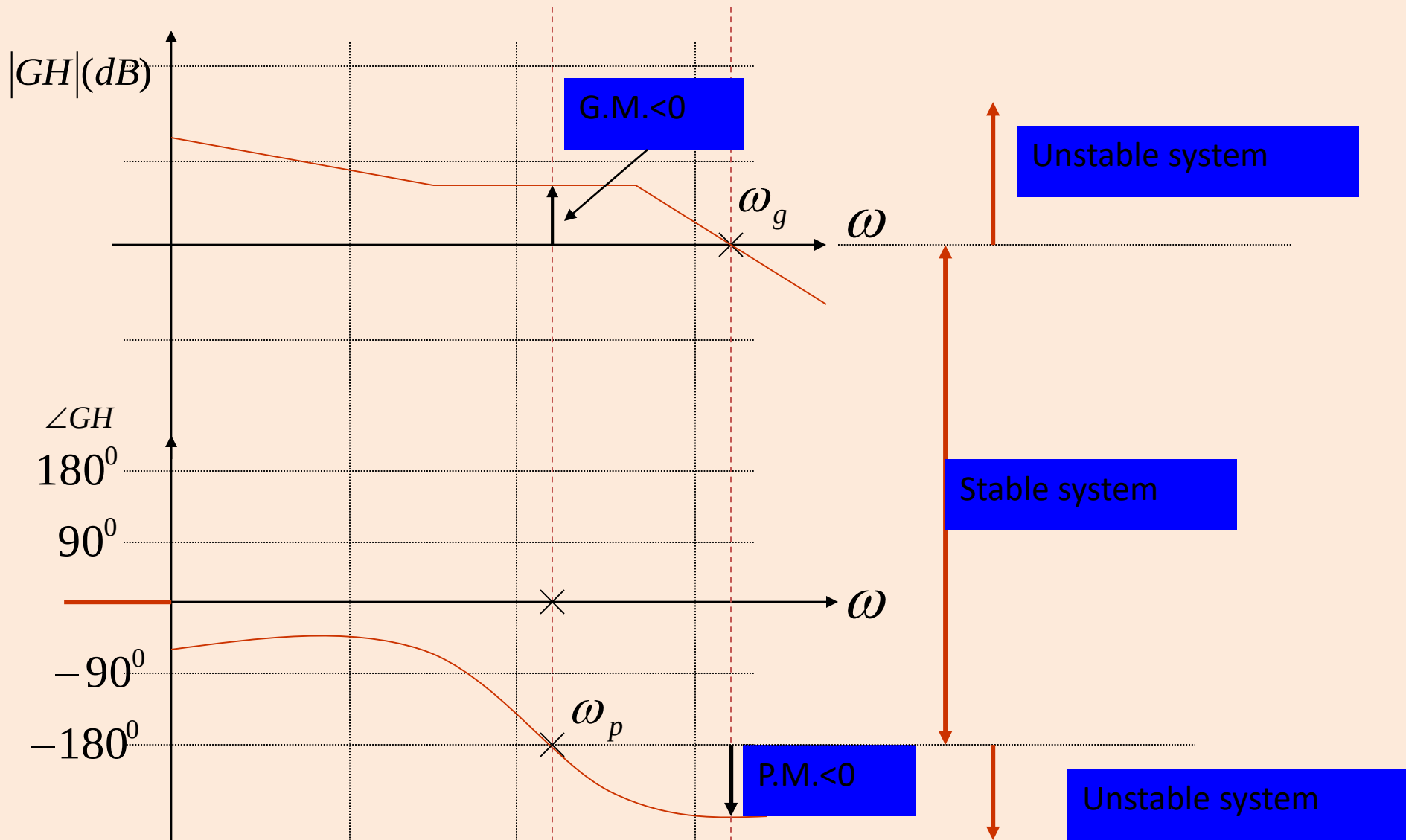
The **phase margin (PM)** is the distance from -180 up to the phase at the gain crossover frequency. **$PM = 180 + \text{phase of } GH \text{ measured at the gain crossover frequency}$**



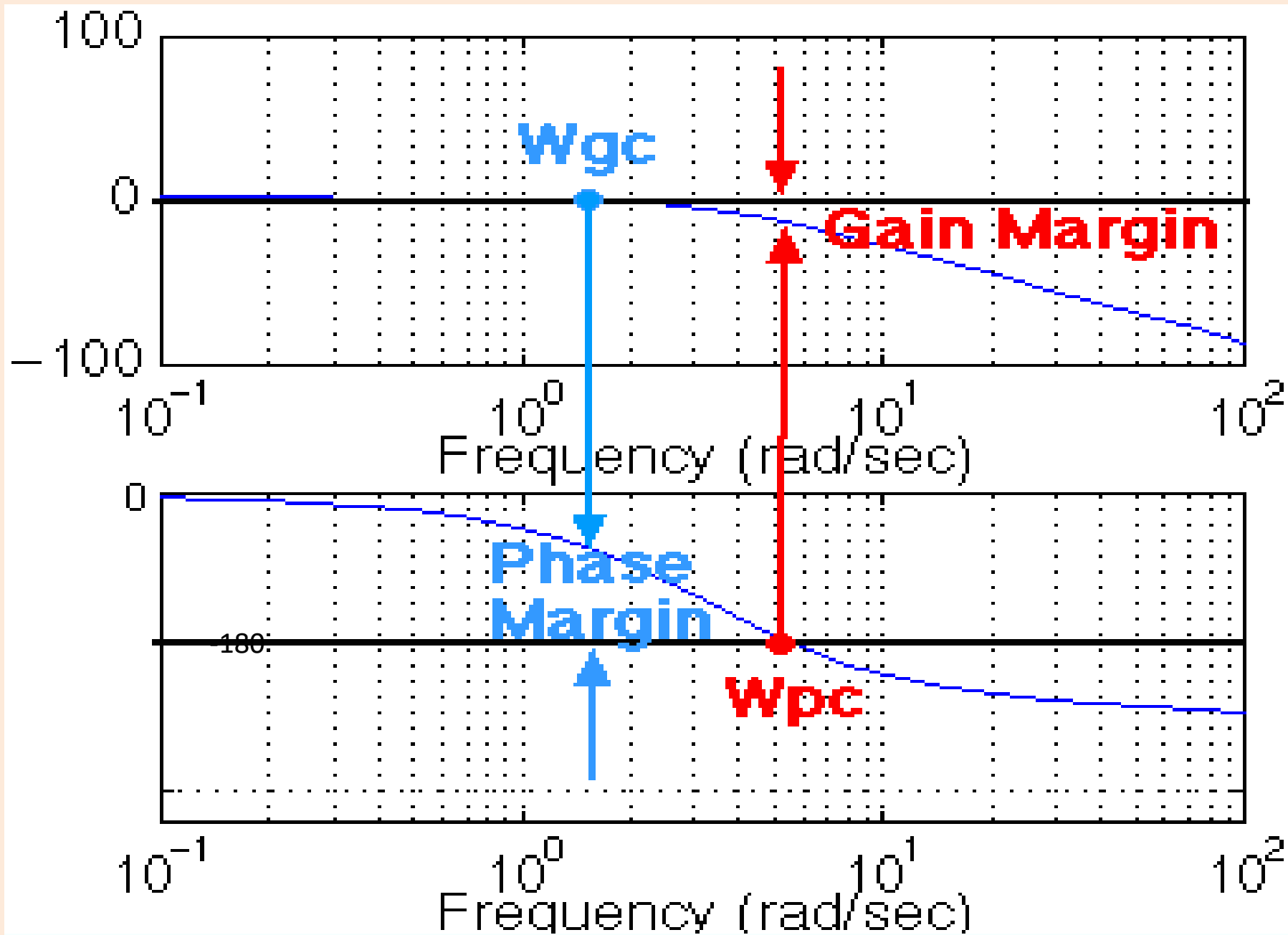
$$(-1, 0) \Rightarrow \begin{cases} 0\text{ dB} \\ -180^\circ \end{cases}$$

Gain crossover frequency: ω_g

phase crossover frequency: ω_p



Gain and Phase Margin



THANKS