



COMPLEX MATRICES

Now that you are able to add, subtract, and multiply complex numbers, you can apply these operations to matrices whose entries are complex numbers. Such a matrix is called **complex**.

Definition of a Complex Matrix

A matrix whose entries are complex numbers is called a **complex matrix**.

Operations with Complex Matrices

Let A and B be the complex matrices

$$A = \begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2i & 0 \\ i & 1+2i \end{bmatrix}$$

and determine each of the following.

- a. $3A$ b. $(2-i)B$ c. $A+B$ d. BA

Solution:

$$\text{a. } 3A = 3 \begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix} = \begin{bmatrix} 3i & 3+3i \\ 6-9i & 12 \end{bmatrix}$$

$$\text{b. } (2-i)B = (2-i) \begin{bmatrix} 2i & 0 \\ i & 1+2i \end{bmatrix} = \begin{bmatrix} 2+4i & 0 \\ 1+2i & 4+3i \end{bmatrix}$$

$$\text{c. } A+B = \begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix} + \begin{bmatrix} 2i & 0 \\ i & 1+2i \end{bmatrix} = \begin{bmatrix} 3i & 1+i \\ 2-2i & 5+2i \end{bmatrix}$$

$$\begin{aligned} \text{d. } BA &= \begin{bmatrix} 2i & 0 \\ i & 1+2i \end{bmatrix} \begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix} \\ &= \begin{bmatrix} -2+0 & 2i-2+0 \\ -1+2-3i+4i+6 & i-1+4+8i \end{bmatrix} \\ &= \begin{bmatrix} -2 & -2+2i \\ 7+i & 3+9i \end{bmatrix} \end{aligned}$$



Finding the Determinant of a Complex Matrix

Find the determinant of the matrix

$$A = \begin{bmatrix} 2 - 4i & 2 \\ 3 & 5 - 3i \end{bmatrix}.$$

Solution:

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 - 4i & 2 \\ 3 & 5 - 3i \end{vmatrix} \\ &= (2 - 4i)(5 - 3i) - (2)(3) \\ &= 10 - 20i - 6i - 12 - 6 \\ &= -8 - 26i \end{aligned}$$

Finding the Inverse of a Complex Matrix

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 - i & -5 + 2i \\ 3 - i & -6 + 2i \end{bmatrix}$$

and verify your solution by showing that $AA^{-1} = I_2$.

Solution:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} -6 + 2i & 5 - 2i \\ -3 + i & 2 - i \end{bmatrix}.$$

Furthermore, because

$$\begin{aligned} |A| &= (2 - i)(-6 + 2i) - (-5 + 2i)(3 - i) \\ &= (-12 + 6i + 4i + 2) - (-15 + 6i + 5i + 2) \\ &= 3 - i \end{aligned}$$

it follows that

$$A^{-1} = \frac{1}{3 - i} \begin{bmatrix} -6 + 2i & 5 - 2i \\ -3 + i & 2 - i \end{bmatrix}$$



$$= \frac{1}{3-i} \left(\frac{1}{3+i} \right) \begin{bmatrix} (-6+2i)(3+i) & (5-2i)(3+i) \\ (-3+i)(3+i) & (2-i)(3+i) \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} -20 & 17-i \\ -10 & 7-i \end{bmatrix}.$$

To verify your solution, multiply A and A^{-1} as follows.

$$AA^{-1} = \begin{bmatrix} 2-i & -5+2i \\ 3-i & -6+2i \end{bmatrix} \frac{1}{10} \begin{bmatrix} -20 & 17-i \\ -10 & 7-i \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Conjugate Transpose of a Matrix

Definition of the Conjugate Transpose of a Complex Matrix

The **conjugate transpose** of a complex matrix A , denoted by A^* , is given by

$$A^* = A^T$$

where the entries of A are the complex conjugates of the corresponding entries of A .

Determine A^* for the matrix

$$A = \begin{bmatrix} 3+7i & 0 \\ 2i & 4-i \end{bmatrix}.$$

Solution:

$$\overline{A} = \begin{bmatrix} \overline{3+7i} & \overline{0} \\ \overline{2i} & \overline{4-i} \end{bmatrix} = \begin{bmatrix} 3-7i & 0 \\ -2i & 4+i \end{bmatrix}$$

$$A^* = \overline{A}^T = \begin{bmatrix} 3-7i & -2i \\ 0 & 4+i \end{bmatrix}$$

Properties of the Conjugate Transpose

If A and B are complex matrices and k is a complex number, then the following properties are true.

1. $(A^*)^* = A$
2. $(A+B)^* = A^* + B^*$
3. $(kA)^* = kA^*$
4. $(AB)^* = B^*A^*$



Unitary Matrices

Recall that a real matrix A is **orthogonal** if and only if $A^{-1} = A^T$. In the complex system, matrices having the property that $A^{-1} = A^*$ are more useful, and such matrices are called **unitary**.

Definition of Unitary Matrix

A complex matrix A is **unitary** when

$$A^{-1} = A^*.$$

Example:

Show that the matrix A is unitary.

$$A = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

Begin by finding the product AA^* .

$$\begin{aligned} AA^* &= \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Solution:

Because

$$AA^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

it follows that $A^* = A^{-1}$. So, A is a unitary matrix.

Definition of a Hermitian Matrix

A square matrix A is **Hermitian** when $A = A^*$.

Examples:



Which matrices are Hermitian?

a. $\begin{bmatrix} 1 & 3-i \\ 3+i & i \end{bmatrix}$

b. $\begin{bmatrix} 0 & 3-2i \\ 3-2i & 4 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$

d. $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix}$

- a. This matrix is not Hermitian because it has an imaginary entry on its main diagonal.
- b. This matrix is symmetric but not Hermitian because the entry in the first row and second column is not the complex conjugate of the entry in the second row and first column.
- c. This matrix is Hermitian.
- d. This matrix is Hermitian because all real symmetric matrices are Hermitian.

Skew-Hermitian matrix. A square matrix such that $A = -A^*$.

$$a_{ij} = -\overline{a_{ji}}$$

for all elements a_{ij} of the matrix. The diagonal elements are either zeros or pure imaginaries.

Example.

$$\begin{bmatrix} i & 1-i & 2 \\ -1-i & 3i & i \\ -2 & i & 0 \end{bmatrix}$$



Linear Systems of Equations

Definition

A system of linear equations is one which may be written in the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \quad (1)$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \quad (2)$$

$$\vdots$$
$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \quad (m)$$

For example, for the following system:

$$3x + 2y - z = 3$$

$$x - y + 2z = 4$$

$$2x + 3y - z = 3$$

$$\text{Augmented Matrix} \longrightarrow \left[\begin{array}{ccc|c} 3 & 2 & -1 & 3 \\ 1 & -1 & 2 & 4 \\ 2 & 3 & -1 & 3 \end{array} \right]$$

-When given a system of equations, to write in augmented matrix form, the coefficients of each variable must be taken and put in a matrix.

Note: A system of linear equations is called **consistent** if it has at least one solution. A system with no solutions is called **inconsistent**.

A system of linear equations with real coefficients has either

- ① a unique solution (a consistent system) or
- ② infinitely many solutions (a consistent system) or
- ③ no solutions (an inconsistent system).



Definition

The following **elementary row operations** can be performed on a matrix:

- (1) Interchange two rows.
- (2) Multiply a row by a nonzero constant.
- (3) Add a multiple of a row to another row.

Gaussian Elimination with Back substitution

$$\begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ 0 & 0 & 0 & \dots & 1 & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

Gaussian Jordan

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

Gaussian Elimination

- (1) Write the augmented matrix of the system of linear equations.
- (2) Use elementary row operations to reduce the augmented matrix to row echelon form.
- (3) Using back substitution, solve the equivalent system that corresponds to the row-reduced matrix.

Example: Solve the following system of equations using Gaussian Elimination:

$$\begin{cases} y + z - 2w = -3 \\ x + 2y - z = 2 \\ 2x + 4y + z - 3w = -2 \\ x - 4y - 7z - w = -19 \end{cases}$$



Solution:

$$\begin{bmatrix} 0 & 1 & 1 & -2 & \vdots & -3 \\ 1 & 2 & -1 & 0 & \vdots & 2 \\ 2 & 4 & 1 & -3 & \vdots & -2 \\ 1 & -4 & -7 & -1 & \vdots & -19 \end{bmatrix}$$

Write augmented matrix.

$$\begin{matrix} \curvearrowright R_2 \\ R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 2 & 4 & 1 & -3 & \vdots & -2 \\ 1 & -4 & -7 & -1 & \vdots & -19 \end{bmatrix}$$

Interchange R_1 and R_2 so first column has leading 1 in upper left corner.

$$\begin{matrix} -2R_1 + R_3 \rightarrow \\ -R_1 + R_4 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & -6 & -6 & -1 & \vdots & -21 \end{bmatrix}$$

Perform operations on R_3 and R_4 so first column has zeros below its leading 1.

$$6R_2 + R_4 \rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & 0 & 0 & -13 & \vdots & -39 \end{bmatrix}$$

Perform operations on R_4 so second column has zeros below its leading 1.

$$\begin{matrix} \frac{1}{3}R_3 \rightarrow \\ -\frac{1}{13}R_4 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 1 & -1 & \vdots & -2 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$$

Perform operations on R_3 and R_4 so third and fourth columns have leading 1's.



A second method of elimination, called **Gauss-Jordan elimination**, continues the reduction process until a *reduced* row-echelon form is obtained.

Example:

Use Gauss-Jordan elimination to solve the system

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

Solution:

The row-echelon form of the linear system can be obtained using the Gaussian elimination.

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

Now, apply elementary row operations until you obtain zeros above each of the leading 1's, as follows.

$$2R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 9 & \vdots & 19 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \quad \begin{array}{l} \text{Perform operations on } R_1 \\ \text{so second column has a} \\ \text{zero above its leading 1.} \end{array}$$
$$\begin{array}{l} -9R_3 + R_1 \rightarrow \\ -3R_3 + R_2 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \quad \begin{array}{l} \text{Perform operations on } R_1 \\ \text{and } R_2 \text{ so third column has} \\ \text{zeros above its leading 1.} \end{array}$$

The matrix is now in reduced row-echelon form. Converting back to a system of linear equations, you have

$$\begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}$$



Example:

Solve the system

$$2x_2 + 3x_3 = 8 \quad 2x_1 + 3x_2 + x_3 = 5, \quad x_1 - x_2 - 2x_3 = -5.$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{array} \right].$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right].$$

back substitution gives the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

Example:

Solve the system

$$w - 2x + y - z = 2$$

$$w - x + y - 2z = 5$$

$$-2w + 2x - 2y + 4z = -10$$

The augmented matrix will be

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 2 \\ 1 & -1 & 1 & -2 & 5 \\ -2 & 2 & -2 & 4 & -10 \end{array} \right]$$



$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & -2 & 0 & 2 & -6 \end{array} \right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The linear system associated with this augmented matrix is

$$w - 2x + y - z = 2, \quad x - z = 3$$

this will have infinitely many solutions, if we express the variables related to the leading entries of the matrix (the **leading variables**) in terms of the other variables (the **free variables**).

Substituting $x = z + 3$ into the equation for w , if we treat the free variables as parameters, the solution may be expressed in vector form as

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

This second example will showcase the fact that the free variables are just the variables that aren't leading variables. The number of leading coefficients is the number of nonzero rows in the row echelon form of the coefficient matrix, we will be able to predict the number of free variables before the solution is determined using back substitution.

Definition

The rank of a matrix is the number of nonzero rows in its row echelon form.

We will call the rank of a matrix A by $rank(A)$, in the first example, the rank was 3, and in the second example the rank will be two (by construction).



Example

Consider the matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{pmatrix}.$$

The *row reduced echelon form* of the above is

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example:

Solve the system

$$x_1 - x_2 + 2x_3 = 3$$

$$x_1 + 2x_2 - x_3 = -3$$

$$2x_2 - 2x_3 = 1$$

The augmented matrix is now

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & -2 & 1 \end{array} \right].$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 5 \end{array} \right]$$



this matrix is inconsistent as it is giving a clear contradiction $0 = 5$, it has no solutions.

Homogeneous Systems. So far we have argued that every system of linear equations has either no solution, a unique solution or infinitely many solutions. There is a special type of system that always has at least one solution.

$$AX = 0$$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

$$b_1x_1 + b_2x_2 + \cdots + b_nx_n = 0$$

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$$

$$\cdot \quad \quad \quad \cdot$$

$$\cdot \quad \quad \quad \cdot$$

To solve we use Gaussian or Gauss- Jordan Elimination

Example:

The augmented matrix of a homogeneous linear system is

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -2 & 0 & 0 \\ 3 & -4 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & -8 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -8 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(backward substitution) The third row only yields $0 = 0$, from the second row we get $x_2 = x_3$, from the first row, we then get $x_1 + 2x_3 - 3x_3 = 0 \Leftrightarrow x_1 = x_3$.



FINDING EIGENVALUES AND EIGENVECTORS

- To do this, we find the values of λ which satisfy the **characteristic equation** of the matrix A , namely those values of λ for which

$$\det(A - \lambda I) = 0,$$

- **STEP 1:** For each eigenvalue λ , we have

$$(A - \lambda I)x = 0,$$

where x is the **eigenvector** associated with **eigenvalue** λ .

- **STEP 2:** Find x by Gaussian elimination. That is, convert the augmented matrix

$$\left(A - \lambda I : 0 \right)$$

to row echelon form, and solve the resulting linear system by back substitution.

We find the **eigenvectors** associated with each of the **eigenvalues**

Example

$$\begin{aligned} \text{Let } A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}. \text{ Then } p(\lambda) &= \det \begin{bmatrix} 2 - \lambda & -4 \\ -1 & -1 - \lambda \end{bmatrix} \\ &= (2 - \lambda)(-1 - \lambda) - (-4)(-1) \\ &= \lambda^2 - \lambda - 6 \\ &= (\lambda - 3)(\lambda + 2). \end{aligned}$$

Example Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$



- To do this, we find the values of λ which satisfy the **characteristic equation** of the matrix A , namely those values of λ for which

$$\det(A - \lambda I) = 0,$$

where I is the 3×3 identity matrix.

- Form the matrix $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 - \lambda & -3 & 3 \\ 3 & -5 - \lambda & 3 \\ 6 & -6 & 4 - \lambda \end{pmatrix}.$$

- Calculate $\det(A - \lambda I)$:

$$\begin{aligned} \det(A - \lambda I) &= (1 - \lambda) \begin{vmatrix} -5 - \lambda & 3 \\ -6 & 4 - \lambda \end{vmatrix} - (-3) \begin{vmatrix} 3 & 3 \\ 6 & 4 - \lambda \end{vmatrix} + 3 \begin{vmatrix} 3 & -5 - \lambda \\ 6 & -6 \end{vmatrix} \\ &= (1 - \lambda) ((-5 - \lambda)(4 - \lambda) - (3)(-6)) + 3(3(4 - \lambda) - 3 \times 6) + 3(3 \times (-6) - (-5 - \lambda)6) \\ &= (1 - \lambda)(-20 + 5\lambda - 4\lambda + \lambda^2 + 18) + 3(12 - 3\lambda - 18) + 3(-18 + 30 + 6\lambda) \\ &= (1 - \lambda)(-2 + \lambda + \lambda^2) + 3(-6 - 3\lambda) + 3(12 + 6\lambda) \\ &= -2 + \lambda + \lambda^2 + 2\lambda - \lambda^2 - \lambda^3 - 18 - 9\lambda + 36 + 18\lambda \\ &= 16 + 12\lambda - \lambda^3. \end{aligned}$$

Notice that this matrix is just equal to A with λ subtracted from each entry on the main diagonal.

To find solutions to $\det(A - \lambda I) = 0$ i.e., to solve

$$\lambda^3 - 12\lambda - 16 = 0.$$



- * Such solutions divide the constant term (-16). The list of possible integer solutions is

$$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16.$$

- * Taking $\lambda = 4$, we find that $4^3 - 12 \cdot 4 - 16 = 0$.
- * Now factor out $\lambda - 4$:

$$(\lambda - 4)(\lambda^2 + 4\lambda + 4) = \lambda^3 - 12\lambda^2 + 16.$$

- * Solving $\lambda^2 + 4\lambda + 4$ by formula¹ gives

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 4}}{2} = \frac{-4 \pm 0}{2},$$

and so $\lambda = -2$ (a repeated root).

- Therefore, the eigenvalues of A are $\lambda = 4, -2$. ($\lambda = -2$ is a repeated root of the characteristic equation.)
- Once the eigenvalues of a matrix (A) have been found, we can find the eigenvectors by Gaussian Elimination.
- **STEP 1:** For each eigenvalue λ , we have

$$(A - \lambda I)x = 0,$$

where x is the eigenvector associated with eigenvalue λ .

- **STEP 2:** Find x by Gaussian elimination. That is, convert the augmented matrix

$$(A - \lambda I : 0)$$

to row echelon form, and solve the resulting linear system by back substitution.

We find the eigenvectors associated with each of the eigenvalues

- Case 1: $\lambda = 4$



– First, form the matrix $A - 4I$:

$$A - 4I = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}.$$

– Construct the augmented matrix $(A - \lambda I : 0)$ and convert it to row echelon form

$$\begin{pmatrix} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix}$$

$$R1 \rightarrow -1/3 \times R1 \rightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} R2 \rightarrow R2 - 3 \times R1 \\ R3 \rightarrow R3 - 6 \times R1 \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{pmatrix}$$

$$R2 \rightarrow -1/12 \times R2 \rightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & -12 & 6 & 0 \end{pmatrix}$$

$$R3 \rightarrow R3 + 12 \times R2 \rightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R1 \rightarrow R1 - R2 \rightarrow \begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

– Rewriting this augmented matrix as a linear system gives

$$x_1 - 1/2x_3 = 0$$

$$x_2 - 1/2x_3 = 0$$



So the eigenvector $\mathbf{x}$ is given by:

$$\mathbf{x} = \begin{pmatrix} x_1 = \frac{x_3}{2} \\ x_2 = \frac{x_3}{2} \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

For any real number $x_3 \neq 0$. Those are the eigenvectors of A associated with the eigenvalue $\lambda = 4$.

• Case 2: $\lambda = -2$

- We seek vectors $\mathbf{x}$ for which $(A - \lambda I)\mathbf{x} = \mathbf{0}$.
- Form the matrix $A - (-2)I = A + 2I$

$$A + 2I = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}.$$

- Now we construct the augmented matrix $(A - \lambda I : \mathbf{0})$ and convert it to row echelon form

$$\begin{pmatrix} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix}$$

$$\begin{array}{l} R1 \rightarrow 1/3 \times R1 \\ \hline \end{array} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix}$$

$$\begin{array}{l} R2 \rightarrow R2 - 3 \times R1 \\ R3 \rightarrow R3 - 6 \times R1 \\ \hline \end{array} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



– When this augmented matrix is rewritten as a linear system, we obtain

$$x_1 + x_2 - x_3 = 0,$$

so the eigenvectors $\mathbf{x}$ associated with the eigenvalue $\lambda = -2$ are given by:

$$\mathbf{x} = \begin{pmatrix} x_1 = x_3 - x_2 \\ x_2 \\ x_3 \end{pmatrix}$$

– Thus

$$\mathbf{x} = \begin{pmatrix} x_3 - x_2 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{for any } x_2, x_3 \in \mathbb{R} \setminus \{0\}$$

are the eigenvectors of A associated with the eigenvalue $\lambda = -2$.

An example of repeated eigenvalue having only two eigenvectors.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Step 1. Form the Characteristic Equation.

The characteristic equation is:

$$\begin{aligned} \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} &= 0 \\ \lambda(\lambda^2 - (-1)(-1)) - (-1)((-1)\lambda - (-1)(-1)) - 1((-1)\lambda - (-1)(-1)) &= 0 \\ \lambda^3 - 3\lambda^2 - 2 &= 0 \\ (\lambda + 1)^2(\lambda - 2) &= 0 \end{aligned}$$



Find Eigenvectors corresponding to each Eigenvalue:

We now need to find eigenvectors corresponding to each eigenvalue.

case(i) $\lambda_1 = -1$.

The eigenvectors are the solution space of the following system:

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$-x_1 - x_2 - x_3 = 0; \quad x_1 = -x_2 - x_3$$

The set of eigenvectors corresponding to $\lambda_1 = -1$ is,

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1 = -x_2 - x_3 \right\}$$
$$\left\{ \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} \mid \text{at least one of } x_2 \text{ and } x_3 \text{ is a non-zero real number} \right\}$$
$$\left\{ x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid \text{at least one of } x_2 \text{ and } x_3 \text{ is a non-zero real number} \right\}$$

We therefore can get two linearly independent eigenvectors corresponding to $\lambda_1 = -1$:

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$



case(ii) $\lambda_2 = 2$.

The eigenvectors are the solution space of the following system:

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 2x_1 - x_2 - x_3 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

Since this system of equations looks fairly complicated, it may be a good idea to use row reduction to simplify the system.

$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array} \right) \xrightarrow{R_3 \leftrightarrow R_1} \left(\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ -1 & 2 & -1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right)$$

$$R_1 = (-1)R_1 \xrightarrow{\rightarrow} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -1 & 2 & -1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right)$$

$$\begin{array}{l} R_3 = R_2 - 2R_1 \\ R_2 = R_1 + R_2 \end{array} \xrightarrow{\rightarrow} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right)$$

$$R_2 = 1/3 R_2 \xrightarrow{\rightarrow} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right)$$

$$R_3 = R_3 + 3R_2 \xrightarrow{\rightarrow} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The resulting set of equations is:

$$\begin{aligned} x_2 - x_3 &= 0 \implies x_2 = x_3 \\ x_1 + x_2 - 2x_3 &= 0 \implies x_1 = x_3 \end{aligned}$$

The set of eigenvectors corresponding to $\lambda_2 = 2$ is,



$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1 = x_2 = x_3 \right\}$$

$$\left\{ \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} \mid x_3 \text{ is a non-zero real number} \right\}$$

$$\left\{ x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mid x_3 \text{ is a non-zero real number} \right\}$$

An eigenvector corresponding to $\lambda_2 = 2$ is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Example: find the **eigenvalue** and **eigenvectors**

$$\begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$$

$$\begin{aligned} \text{Det} (A - \lambda I_n) &= \text{Det} \begin{pmatrix} 2 - \lambda & -1 \\ 5 & -2 - \lambda \end{pmatrix} \\ &= (2 - \lambda) (-2 - \lambda) - (-1)(5) \\ &= \lambda^2 + 1 \end{aligned}$$

and so the eigenvalues are $\lambda = -i$ & $\lambda = i$.

$$\lambda = -i$$

$$\begin{pmatrix} 2 - (-i) & -1 \\ 5 & -2 - (-i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$



As always, we set up an appropriate augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 2+i & -1 & 0 \\ 5 & -2+i & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & -\frac{2}{5} + \frac{i}{5} & 0 \\ 5 & -2+i & 0 \end{array} \right)$$

$$\frac{-1}{2+i} = \frac{(-1)(2-i)}{(2+i)(2-i)} = \frac{-2+i}{2^2 - (-1)} = -\frac{2}{5} + \frac{i}{5}$$

$$\sim \left(\begin{array}{cc|c} 1 & -\frac{2}{5} + \frac{i}{5} & 0 \\ 0 & 0 & 0 \end{array} \right)$$

Hence, $x_1 = \left(\frac{2}{5} - \frac{i}{5} \right) x_2$ and so $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 2-i \\ 5 \end{pmatrix}$ for all scalars t .

To find the eigenspace corresponding to $\lambda = i$ we must solve

$$\left(\begin{array}{cc} 2-i & -1 \\ 5 & -2-i \end{array} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We again set up an appropriate augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & -\frac{2}{5} - \frac{i}{5} & 0 \\ 5 & -2-i & 0 \end{array} \right)$$

$$\sim \left(\begin{array}{cc|c} 1 & -\frac{2}{5} - \frac{i}{5} & 0 \\ 0 & 0 & 0 \end{array} \right)$$

Hence, $x_1 = \left(\frac{2}{5} + \frac{i}{5} \right) x_2$ and so $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 2+i \\ 5 \end{pmatrix}$ for all scalars t .



Example

$$A = \begin{bmatrix} 3 & -1 & 6 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}, \quad A - \lambda I = \begin{bmatrix} 3 - \lambda & -1 & 6 \\ 0 & -\lambda & 6 \\ 0 & 0 & 2 - \lambda \end{bmatrix}.$$

$$\det(A - \lambda I) = (3 - \lambda)(-\lambda)(2 - \lambda) = 0$$

The eigenvalues are 3, 0, 2, exactly the diagonal elements.

Diagonalization

DEFINITION. A square matrix of size n is called a diagonal matrix if it is of the form

$$\begin{pmatrix} d_1 & 0 & . & . & . & 0 \\ 0 & d_2 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & d_{n-1} & 0 & . \\ 0 & . & . & . & 0 & d_n \end{pmatrix}$$

DEFINITION. Two square matrices A and B of size n are said to be similar, if there exists an invertible matrix P such that

$$B = P^{-1}AP$$

DEFINITION. A matrix A is diagonalizable if A is similar to a diagonal matrix. i.e. there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix. In this case, P is called a diagonalizing matrix.

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}.$$



the eigenvalues of A are: $\lambda_1 = 2$, $\lambda_2 = 5$.

An eigenvector corresponding to $\lambda_1 = 2$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

An eigenvector corresponding to $\lambda_2 = 5$ is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Define a new matrix P such that its columns are the eigenvectors of A .

$$P = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Note that since eigenvectors corresponding to distinct eigenvalues are linearly independent, the columns of P are linearly independent, hence P is an invertible matrix. (We can also confirm this by verifying that the determinant of P is non-zero).

$$P^{-1} = \frac{-1}{3} \begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$\begin{aligned} P^{-1}AP &= \frac{-1}{3} \begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ &= \frac{-1}{3} \begin{pmatrix} 2 & -4 \\ -5 & -5 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ &= \frac{-1}{3} \begin{pmatrix} -6 & 0 \\ 0 & -15 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}, \end{aligned}$$

which is a diagonal matrix. Therefore A is diagonalizable and a diagonalizing matrix for A is

$$P = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix},$$

and the corresponding diagonal matrix D is

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}.$$



$$A = \begin{pmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{pmatrix}.$$

- (i) Find the eigenvalues and eigenvectors of A .
- (ii) Is A diagonalizable?
- (iii) Find a diagonalizing matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix.

SOLUTION: (i) In previous lectures we have found,

The eigenvalues of A are: $\lambda_1 = -1$, $\lambda_2 = -6$ and $\lambda_3 = 5$.

An eigenvector corresponding to $\lambda_1 = -1$ is $\begin{pmatrix} \frac{-1}{5} \\ \frac{-9}{25} \\ 1 \end{pmatrix}$.

An eigenvector corresponding to $\lambda_2 = -6$ is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

An eigenvector corresponding to $\lambda_3 = 5$ is $\begin{pmatrix} 1 \\ \frac{-3}{11} \\ 1 \end{pmatrix}$.

- (ii) The matrix A is diagonalizable, since A is a square matrix of size 3×3 and we have found three linearly independent eigenvectors for A .
- (iii) A diagonalizing matrix P has the eigenvectors as its columns. Let

$$P = \begin{pmatrix} \frac{-1}{5} & 0 & 1 \\ \frac{-9}{25} & 1 & \frac{-3}{11} \\ 1 & 0 & 1 \end{pmatrix}.$$

The corresponding diagonal matrix D will have the eigenvalues as diagonal entries corresponding to the columns of P . i.e. the i th diagonal entry will be the eigenvalue corresponding to the eigenvector in the i th column.

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$



We will now verify that $P^{-1}AP = D$.

$$\begin{aligned}AP &= \begin{pmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-1}{5} & 0 & 1 \\ \frac{-9}{25} & 1 & \frac{-3}{11} \\ 1 & 0 & 1 \end{pmatrix} \\&= \begin{pmatrix} 1/5 & 0 & 5 \\ 9/25 & -6 & -15/11 \\ -1 & 0 & 5 \end{pmatrix} \\PD &= \begin{pmatrix} \frac{-1}{5} & 0 & 1 \\ \frac{-9}{25} & 1 & \frac{-3}{11} \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 5 \end{pmatrix} \\&= \begin{pmatrix} 1/5 & 0 & 5 \\ 9/25 & -6 & -15/11 \\ -1 & 0 & 5 \end{pmatrix} \\AP &= PD\end{aligned}$$

Since the columns of P are linearly independent, P has non-zero determinant and is therefore invertible. We multiply (1) by P^{-1} on both sides.

$$\begin{aligned}P^{-1}AP &= P^{-1}PD, \\&= ID \\&= D\end{aligned}$$