



Vectors:

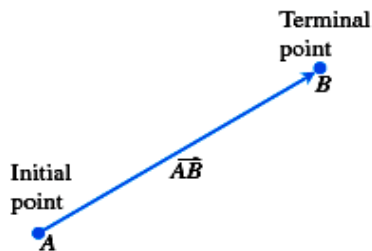
A vector is a matrix that has only one row – then we call the matrix a **row vector** – or only one column – then we call it a **column vector**.

A **row vector** is of the form: $a = [a_1 \ a_2 \ \dots \ a_n]$

A **column vector** is of the form:

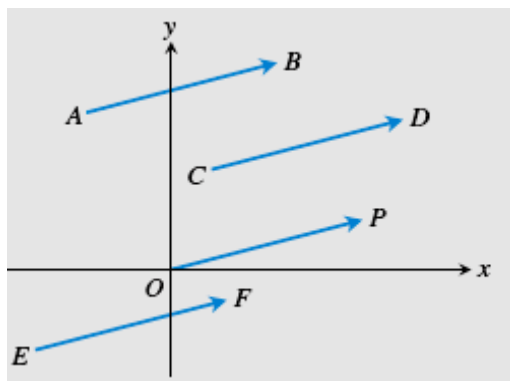
$$b = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

A quantity such as force, displacement, or velocity is called a vector and is represented by a directed line segment



A **vector** in the plane is directed line segment. The directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B; its **length** is denoted by $|\overrightarrow{AB}|$.

Two vectors are **equal** if they have the same length and direction.





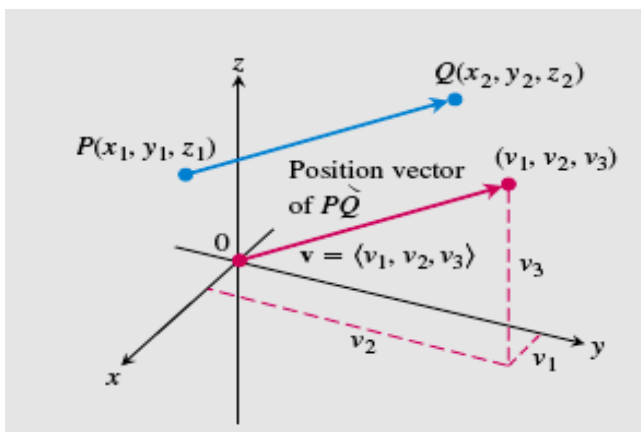
Component form

If $\mathbf{v}$ is a **two dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2) , then the **Component form** of $\mathbf{v}$ is:

$$\mathbf{v} = (v_1, v_2)$$

If $\mathbf{v}$ is a **three dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3) , then the **Component form** of $\mathbf{v}$ is:

$$\mathbf{v} = (v_1, v_2, v_3)$$



The numbers v_1, v_2 and v_3 are called the components of $\mathbf{v}$.

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the standard position vector

$\mathbf{v} = (v_1, v_2, v_3)$ equal to $\overrightarrow{PQ}$ is

$$\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



The only vector with length **0** is the zero vector $0 = (0,0)$ or $0 = (0,0,0)$. This vector is also the only vector with no specific direction.

Example: Find the component form and the length of the vector with initial point $P(-3,4,1)$ and terminal point $Q(-5,2,2)$

Solution:

a) $v = (-5 + 3, 2 - 4, 2 - 1)$

The component form of $\overrightarrow{PQ}$ is $v = (-2, -2, 1)$

b) The length or magnitude of $v = \overrightarrow{PQ}$ is $|v| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3$

Vector Addition and Multiplication of a vector by a scalar

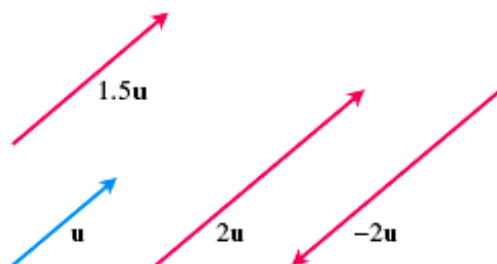
Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be vectors with k a scalar.

Addition:

$$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

Scalar multiplication: $ku = (ku_1, ku_2, ku_3)$

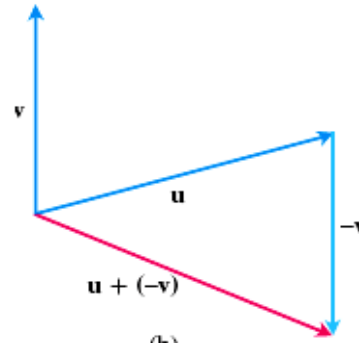
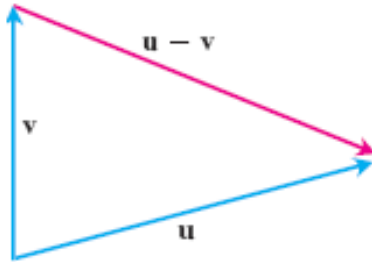
If the length of ku is the absolute value of the scalar k times the length of u . The vector $(-1)u = -u$ has the same length as u but points in the opposite direction.





If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, $u - v = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$

Note that $(u - v) + v = u$ and the difference $u - v$ as the sum $u + (-v)$



Example.: Let $u = (-1, 3, 1)$ and $v = (4, 7, 0)$ vectors, find

- a) $2u + 3v$ b) $u - v$ c) $\left| \frac{1}{2}u \right|$

Solution:

a) $2u + 3v = (-2, 6, 2) + (12, 21, 0) = (10, 27, 2)$

b) $u - v = (-5, -4, 1)$

c) $\left| \frac{1}{2}u \right| = \left| \left(\frac{-1}{2}, \frac{3}{2}, \frac{1}{2} \right) \right| = \frac{1}{2}\sqrt{11}$

Properties of vector operations:

Let u , v and w be vectors and a and b be scalars.

- 1) $u + v = v + u$
- 2) $(u + v) + w = u + (v + w)$
- 3) $u + 0 = u$
- 4) $u + (-u) = 0$
- 5) $0u = 0$
- 6) $1u = u$
- 7) $a(bu) = (ab)u$
- 8) $a(u + v) = au + av$
- 9) $(a + b)u = au + bu$



Unit vectors

A vector v of length 1 is called **unit vector**. The standard unit vectors are:

$$i = (1,0,0) \quad , \quad j = (0,1,0) \quad , \quad k = (0,0,1)$$

$$\begin{aligned} v = (v_1, v_2, v_3) &= (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) \\ &= v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1) \\ &= v_1 i + v_2 j + v_3 k \end{aligned}$$

We call the scalar (or number) v_1 the ***i*-component** of the vector v , v_2 the ***j*-component** of the vector v , and v_3 the ***k*-component**. In component form,

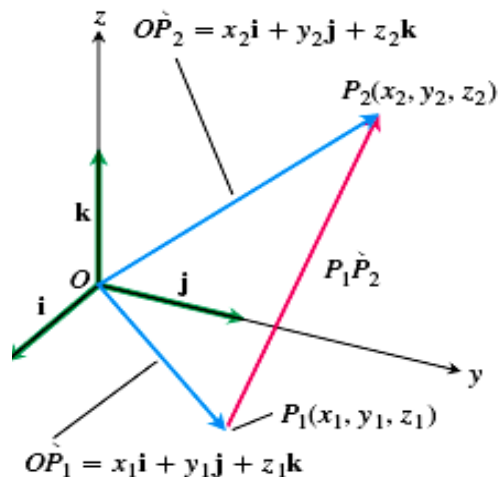
$P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1 P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

If $v \neq 0$, then

$$u = \frac{v}{|v|} \text{ is a unit vector in the}$$

direction of v , called **the direction** of the nonzero vector v .





Example.:

Find a unit vector u in the direction of the points $P_1(1,0,1)$ and $P_2(3,2,0)$.

Solution

$$\overrightarrow{P_1P_2} = (3-1)\mathbf{i} + (2-0)\mathbf{j} + (0-1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$$

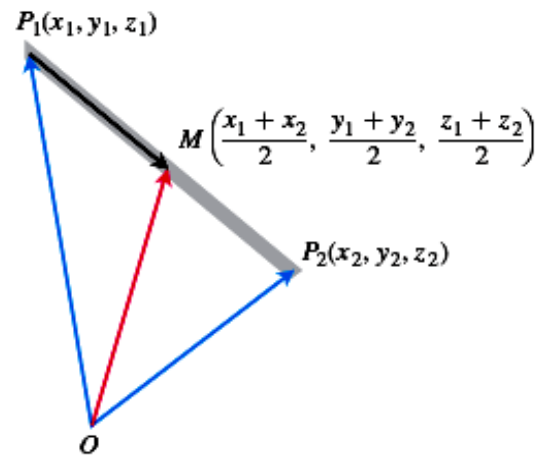
$$u = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$$

The unit vector u is the **direction** of $\overrightarrow{P_1P_2}$.

Midpoint of a line segment

The Midpoint M of a line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$



Example:

The midpoint of the segment joining the points $P_1(3,-2,0)$ and $P_2(7,4,4)$ is

$$\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2} \right) = (5,1,2)$$



Product of vectors

u & v are vectors,

There are two kinds of multiplication of two vectors:

- 1- The scalar product (dot product) $u \cdot v$. The result is a **scalar**.
- 2- The vector product (cross product) $u \times v$. The result is a **vector**

1) The dot product

In this section, we show how to calculate easily the angle between two vectors directly from their components. The dot product is also called **inner** or **scalar** products because the product results in scalar, not a vector.

Def.: The dot product $u \cdot v$ of vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is:

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Note:

$$\begin{bmatrix} i \cdot i \\ j \cdot j \\ k \cdot k \end{bmatrix} = 1.1 = 1, \quad \begin{bmatrix} i \cdot j \\ j \cdot k \\ k \cdot j \end{bmatrix} = 0$$

Example:

a)

$$(3, 5) \cdot (-1, 2) = 3(-1) + 5(2) = 7 \quad \text{scalar}$$

$$(3i + 5j) \cdot (-i + 2j) = 7$$

b)

$$(1, -3, 4) \cdot (1, 5, 2) = 1 - 15 + 8 = -6 \quad \text{scalar}$$

$$(i - 3j + 4k) \cdot (i + 5j + 2k) = -6$$



Angle between two vectors

The angle θ between two nonzero vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is given by $\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos \theta$

$$\theta = \cos^{-1} \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} \right) \quad \text{where } \theta \quad (0 \leq \theta \leq \pi)$$

Example: Find the angle between two vectors

$\vec{u} = 2\vec{i} - \vec{j} + 2\vec{k}$, $\vec{v} = \vec{i} - 2\vec{j} + 2\vec{k}$ in space

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \frac{2 + 2 + 4}{\sqrt{4 + 1 + 4} \cdot \sqrt{1 + 4 + 4}}$$

$$\cos \theta = \frac{8}{9} \Rightarrow \theta = \cos^{-1} \frac{8}{9}$$

Example:

Find the angle θ in the triangle ACB determined by the vertices:

$A = (0,0)$, $B(3,5)$ and $C(5,2)$

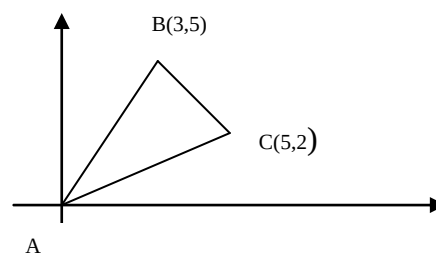
$$\vec{CA} = (-5, -2) \quad \text{and} \quad \vec{CB} = (-2, 3)$$

$$\vec{CA} \cdot \vec{CB} = (-5)(-2) + (-2)(3) = 4$$

$$|\vec{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

$$|\vec{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

$$\theta = \cos^{-1} \left(\frac{4}{\sqrt{29} \cdot \sqrt{13}} \right)$$





Orthogonal vectors

Vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are **orthogonal** (or **perpendicular**) if and only if $u \cdot v = 0$

Example:

a) $u = (3, -2)$ and $v = (4, 6)$ are orthogonal because $u \cdot v = 0$

b) $u = 3i - 2j + k$ and $v = 2j + 4k$ are orthogonal because $u \cdot v = 0$

c) $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ since

$$\begin{aligned} \mathbf{0} \cdot u &= (0, 0, 0) \cdot (u_1, u_2, u_3) \\ &= 0 \end{aligned}$$

Properties of the Dot product

If u, v and w are any vectors and c is a scalar, then

1) $u \cdot v = v \cdot u$

2) $(cu) \cdot v = u \cdot (cv) = c(u \cdot v)$

3) $u \cdot (v + w) = u \cdot v + u \cdot w$

4) $u \cdot u = |u|^2$

5) $\mathbf{0} \cdot u = 0$

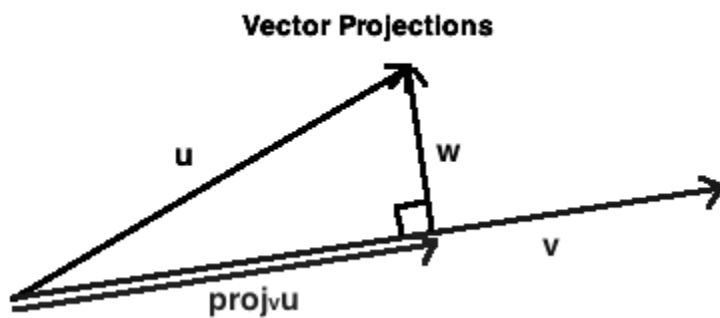
Vector projection

Definition: Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then the Vector Projection of $\vec{u}$ Along $\vec{v}$ denoted

$$\vec{u}_{\vec{v}} = \text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

$\vec{u}_{\vec{v}}$ is a scalar multiple of $\vec{v}$ such that for some

vector $\vec{w} \in \mathbb{R}^n$ such that $\text{proj}_{\vec{v}} \vec{u} \perp \vec{w}$ we have that $\text{proj}_{\vec{v}} \vec{u} + \vec{w} = \vec{u}$.



Example: Find the vector projection of $u = 6i + 3j + 2k$ onto $v = i - 2j - 2k$

Solution:

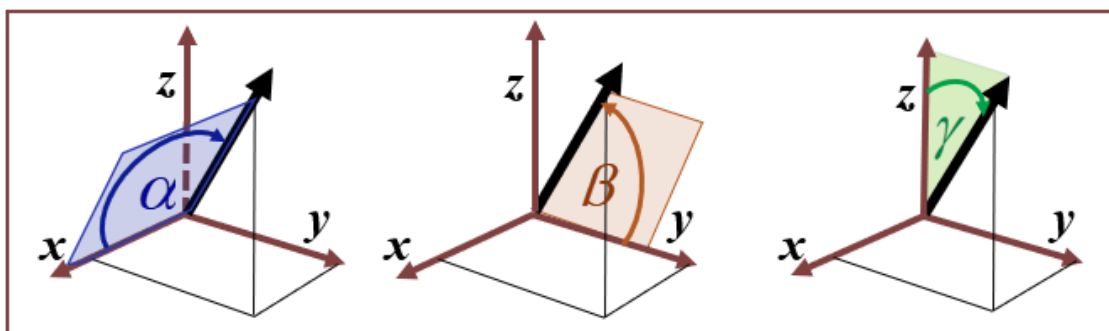
$$\text{proj}_v u = \left(\frac{u \cdot v}{|v|^2} \right) v = \frac{u \cdot v}{v \cdot v} v = \frac{6 - 6 - 4}{1 + 4 + 4} (i - 2j - 2k) = \frac{-4}{9} (i - 2j - 2k) = \frac{-4}{9} i + \frac{8}{9} j + \frac{8}{9} k$$

direction angles

α = angle between v and i (positive x -axis)

β = angle between v and j (positive y -axis)

γ = angle between v and k (positive z -axis)



For the vector $v = ai + bj + ck$, the direction angles α , β , and γ are:

$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{a}{\|v\|}, \quad \cos \beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{b}{\|v\|}, \quad \cos \gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{c}{\|v\|}$$

It also turns out that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$



Example: Find the **direction angles** of the following vector $v = -6i + 3j - 2k$

$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{a}{\sqrt{(-6)^2 + 3^2 + (-2)^2}} = \frac{a}{\|v\|} = \frac{-6}{7}; \quad \alpha = 149^\circ$$

$$\cos \beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{b}{\sqrt{(-6)^2 + 3^2 + (-2)^2}} = \frac{b}{\|v\|} = \frac{3}{7}; \quad \beta = 64.6^\circ$$

$$\cos \gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{c}{\sqrt{(-6)^2 + 3^2 + (-2)^2}} = \frac{c}{\|v\|} = \frac{-2}{7}; \quad \gamma = 106.6^\circ$$

Homework:

1) Let $u = (3, -2)$ and $v = (-2, 5)$ vectors. Find the component form and magnitude (length) of the vector.

1. $-2u + 5v$

2. $\frac{3}{5}u + \frac{4}{5}v$

2) Find the component form of the vector:

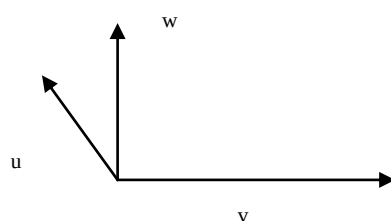
a. The vector $\overrightarrow{PQ}$ where $P = (1, 3)$ and $Q = (2, -1)$.

b. The vector $\overrightarrow{OP}$ where O is the origin and P is the midpoint of segment RS , where $R = (2, -1)$ and $S = (-4, 3)$.

c. The vector from the point $A = (2, 3)$ to the origin.

d. The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$, where $A = (1, -1)$, $B = (2, 0)$, $C = (-1, 3)$ and $D = (-2, 2)$

3) Let v , u and w as in the figure: find a) $u + v$, b) $u + v + w$, c) $u - v$ and d) $u - w$





4) Find the direction of $\overrightarrow{P_1P_2}$ and the midpoint of line segment P_1P_2 .

a. $P_1(-1,1,5)$ and $P_2(2,5,0)$

b. $P_1(0,0,0)$ and $P_2(2,-2,-2)$

5) Find $v \cdot u$, $|v|$, $|u|$, the cosine of the angle between v and u , the scalar component of u in the direction of v and the vector $\text{proj}_v u$.

a) $v = 2i - 4j + \sqrt{5}k$, $u = -2i + 4j - \sqrt{5}k$

b) $v = (\frac{3}{5})i + (\frac{4}{5})k$, $u = 5i + 12j$

6) Find the angles between the vectors:

a) $u = 2i - 2j + k$, $v = 3i + 4k$

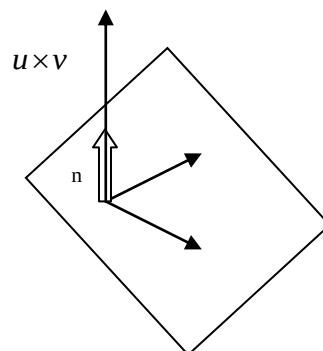
2) The Cross product

The cross product is also called **vector** product because the product results a vector.

Def.: The cross product $u \times v = (|u||v| \sin \theta) \mathbf{n}$, $\mathbf{n}$ unit vector (**normal**) **perpendicular to the plane.**
the area A of the parallelogram

$$A = |u||v| \sin \theta = |u \times v|.$$

Note: The vector $u \times v$ is orthogonal to both u and v





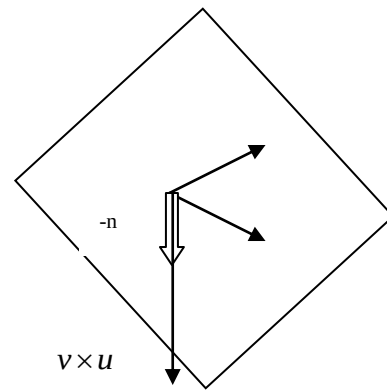
Parallel vectors

Nonzero vectors u and v are parallel if and only if $u \times v = 0$.

Properties of the cross product

If u , v and w are any vectors and r , s are scalars, then

- 1) $(ru) \times (sv) = (rs)(v \times u)$
- 2) $u \times (v + w) = u \times v + u \times w$
- 3) $(v + w) \times u = v \times u + w \times u$
- 4) $v \times u = -(u \times v)$
- 5) $0 \times u = 0$



Notes:

$$\begin{aligned} i \times j &= -(j \times i) = k \\ j \times k &= -(k \times j) = i \\ k \times i &= -(i \times k) = j \end{aligned}$$

$$\left[\begin{array}{l} i \times i \\ j \times j \\ k \times k \end{array} \right] = 0$$

Calculating Cross product using determinants

If $u = u_1i + u_2j + u_3k$ and $v = v_1i + v_2j + v_3k$, then

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example:

Find $u \times v$ and $v \times u$ if $u = 2i + j + k$ and $v = -4i + 3j + k$



Solution

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$

$$= -2i - 6j + 10k$$

$$v \times u = -(u \times v) = 2i + 6j - 10k$$

Example: Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

Solution

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors.

$$\overrightarrow{PQ} = (2-1)i + (1+1)j + (-1-0)k = i + 2j - k$$

$$\overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)k = -2i + 2j + 2k$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} k$$

$$= 6i + 6k$$

Example: Find a unit vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

Solution

Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, its direction $\mathbf{n}$ is a unit vector perpendicular to the plane

$$n = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6i + 6k}{6\sqrt{2}} = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}k$$



Example We will compute the area of the triangle whose vertices are the points

$$P_1 = (1, 1, 0), P_2 = (3, 0, 4) \text{ and } P_3 = (0, 5, 2).$$

This triangle is half of a parallelogram with adjacent sides

$$\mathbf{u} = P_1P_2 = \langle 3 - 1, 0 - 1, 4 - 0 \rangle = \langle 3, -1, 4 \rangle,$$

$$\mathbf{v} = P_1P_3 = \langle 0 - 1, 5 - 1, 2 - 0 \rangle = \langle -1, 4, 2 \rangle.$$

The area of this parallelogram is

$$\begin{aligned} |\mathbf{u} \times \mathbf{v}| &= |\langle -1(2) - 4(4), 4(-1) - 3(2), 3(4) - (-1)(-1) \rangle| \\ &= |\langle -18, -10, 11 \rangle| = \sqrt{545} \approx 23.345. \end{aligned}$$

Therefore, the area of the triangle is $\sqrt{545}/2 \approx 11.673$.

Calculating the Triple scalar product (volume): also called **Box product**

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example: Find the volume of the box determined by $\mathbf{u} = i + 2j - k$, $\mathbf{v} = -2i + 3k$ and $\mathbf{w} = 7j - 4k$.

Solution

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = -23$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed.

- The triple product of three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, given by $|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$, is the volume of the parallelepiped defined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.