



Vector Functions:

A function $\mathbf{r} : \mathbb{R} \rightarrow V_n$ is called a vector-valued function.

That is, the input is a real number and the output is a vector.

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

We call $f(t), g(t), h(t)$ the component functions of $\mathbf{r}$.

Position, Velocity, Acceleration and speed Vectors.

Let $\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ the position of a particle moving along a curve then the **velocity** is the derivative of $\mathbf{r}$

$$\mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

The **acceleration** is $\mathbf{a}(t)$

$$\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$

The magnitude of the velocity vector = **speed** = $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Example: Find the position, velocity, speed and acceleration the given value of t .

a) $\mathbf{r}(t) = t^2\mathbf{i} - t^3\mathbf{j}$ at $t=2$

Position: $\mathbf{r}(2) = 4\mathbf{i} - 8\mathbf{j}$

velocity: $\mathbf{v}(t) = 2t\mathbf{i} - 3t^2\mathbf{j} \rightarrow \mathbf{v}(2) = 4\mathbf{i} - 12\mathbf{j}$

speed: $|\mathbf{v}(2)| = \sqrt{4^2 + (-12)^2} = \sqrt{160} = 4\sqrt{10}$

acceleration: $\mathbf{a}(t) = 2\mathbf{i} - 6t\mathbf{j} \rightarrow \mathbf{a}(2) = 2\mathbf{i} - 12\mathbf{j}$

b) $\mathbf{r}(t) = t\mathbf{i} + 3e^t\mathbf{j} + 4e^t\mathbf{k}$ at $t=0$

Position: $\mathbf{r}(0) = 0\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

velocity: $\mathbf{v}(t) = \mathbf{i} + 3e^t\mathbf{j} + 4e^t\mathbf{k} \rightarrow \mathbf{v}(0) = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

speed: $|\mathbf{v}(0)| = \sqrt{1^2 + 3^2 + 4^2} = \sqrt{26}$

acceleration: $\mathbf{a}(t) = 0\mathbf{i} + 3e^t\mathbf{j} + 4e^t\mathbf{k} \rightarrow \mathbf{a}(2) = 0\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

Scalar acceleration $|\mathbf{a}(0)| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$



The Unit Tangent and Unit Normal Vectors

Let $\mathbf{r}(t)$ be a differentiable vector valued function and $\mathbf{v}(t) = \mathbf{r}'(t)$ be the velocity vector. Then we define the *unit tangent vector* by as the unit vector in the direction of the velocity vector.

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$$

Or

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

For all t , $\mathbf{T}(t)$ gives a vector of magnitude 1 in the direction of motion. Note that $\mathbf{T}(t)$ very likely changes when t changes.

For example

Let

$$\mathbf{r}(t) = t\hat{\mathbf{i}} + e^t\hat{\mathbf{j}} - 3t^2\hat{\mathbf{k}}.$$

Find $\mathbf{T}(t)$ and $\mathbf{T}(0)$.

Solution:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \hat{\mathbf{i}} + e^t\hat{\mathbf{j}} - 6t\hat{\mathbf{k}}$$

and

$$\|\mathbf{v}(t)\| = \sqrt{1 + e^{2t} + 36t^2}.$$

To find the unit tangent vector, we just divide

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\hat{\mathbf{i}} + e^t\hat{\mathbf{j}} - 6t\hat{\mathbf{k}}}{\sqrt{1 + e^{2t} + 36t^2}}.$$

To find $\mathbf{T}(0)$ plug in 0 to get



$$\begin{aligned} T(0) &= \frac{\hat{\mathbf{i}} + e^0 \hat{\mathbf{j}} - 6(0)\hat{\mathbf{k}}}{\sqrt{1 + e^{2(0)} + 36(0)^2}} \\ &= \frac{\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{2}} \end{aligned}$$

The **unit normal vector**

Let $\mathbf{r}(t)$ be a differentiable vector valued function and let $\mathbf{T}(t)$ be the unit tangent vector. Then the *principal unit normal vector* $\mathbf{N}(t)$ is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

Example:

Find the unit normal vector for the vector valued function

$$\mathbf{r}(t) = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}$$

and sketch the curve, the unit tangent and unit normal vectors when $t = 1$.

Solution:

First we find the unit tangent vector

$$\mathbf{T}(t) = \frac{\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}}{\sqrt{1 + 4t^2}}.$$

Now use the quotient rule to find $\mathbf{T}'(t)$

$$\mathbf{T}'(t) = \frac{(1 + 4t^2)^{\frac{1}{2}}(2\hat{\mathbf{j}}) - (\hat{\mathbf{i}} + 2t\hat{\mathbf{j}})4t(1 + 4t^2)^{-\frac{1}{2}}}{1 + 4t^2}.$$

Since the unit vector in the direction of a given vector will be the same after multiplying the vector by a positive scalar, we can simplify by multiplying by the factor

$$(1 + 4t^2)(1 + 4t^2)^{\frac{1}{2}}.$$

$$\begin{aligned} \mathbf{T}'(t)(1 + 4t^2)(1 + 4t^2)^{\frac{1}{2}} &= (1 + 4t^2)(2\hat{\mathbf{j}}) - (\hat{\mathbf{i}} + 2t\hat{\mathbf{j}})4t \\ &= -4t\hat{\mathbf{i}} + 2\hat{\mathbf{j}}. \end{aligned}$$

Now we divide by the magnitude (after first dividing by 2) to get

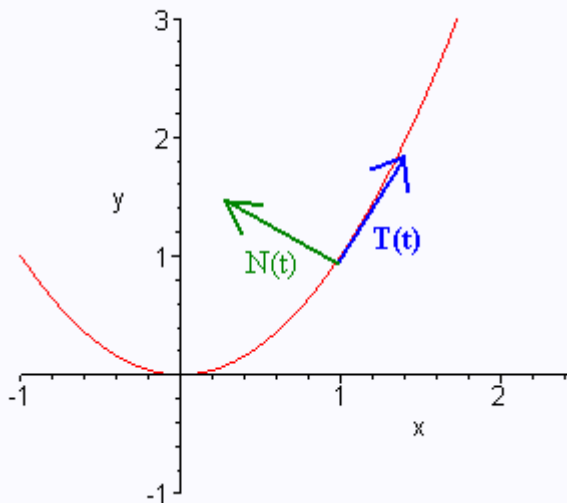


$$\mathbf{N}(t) = \frac{-2t\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{1 + 4t^2}}.$$

Now plug in $t = 1$ for both the unit tangent vector to get

$$\mathbf{T}(1) = \frac{1}{\sqrt{5}}\hat{\mathbf{i}} + \frac{2}{\sqrt{5}}\hat{\mathbf{j}}$$

$$\mathbf{N}(1) = -\frac{2}{\sqrt{5}}\hat{\mathbf{i}} + \frac{1}{\sqrt{5}}\hat{\mathbf{j}}.$$





Example: Find unit vectors tangent and normal to the curve $y = \frac{x^3}{2} + \frac{1}{2}$ at point (1,1).

$$m_1 = y' = \frac{dy}{dx} = \frac{3}{2}x^2 + 0 \quad \text{at point (1,1)}$$

$$m_1 = \frac{3}{2} \text{ tangent}$$

$$m_1 \cdot m_2 = -1$$

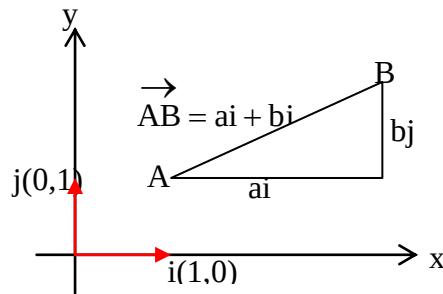
$$m_2 = \frac{-3}{2} = \frac{-a}{b}$$

$$\text{Tangent vector } (\vec{A}) = ai + bj = 2i + 3j$$

$$\text{Normal vector } (\vec{B}) = bi - aj = 3i - 2j$$

$$\text{Unit vector } (\vec{u}) = \frac{\vec{A}}{|\vec{A}|} = \frac{2i + 3j}{\sqrt{4+9}} = \frac{2}{\sqrt{13}}i + \frac{3}{\sqrt{13}}j$$

$$\text{Unit vector } (\vec{u}) = \frac{\vec{B}}{|\vec{B}|} = \frac{3i - 2j}{\sqrt{9+4}} = \frac{3}{\sqrt{13}}i - \frac{2}{\sqrt{13}}j$$



Example:

Find unit vectors tangent and normal to the curve $y = x^2$ at point (2,4).

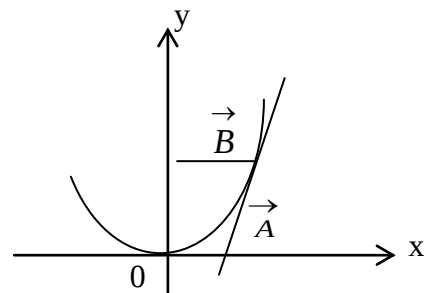
$$m_1 = y' = \frac{dy}{dx} = 2x \quad \text{at point (2,4)}$$

$$\text{Tangent vector } (\vec{A}) = bi + aj = i + 4j$$

$$\text{Unit vector } (\vec{u}) = \frac{\vec{A}}{|\vec{A}|} = \frac{i + 4j}{\sqrt{1+16}} = \frac{1}{\sqrt{17}}i + \frac{4}{\sqrt{17}}j$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow 4 = \frac{4 - 0}{2 - u} \Rightarrow \therefore u = 1$$

$$\vec{P_1P_1} = (1 - 2)i + (0 - 4)j = -i - 4j \quad \text{Vector pointing down word}$$



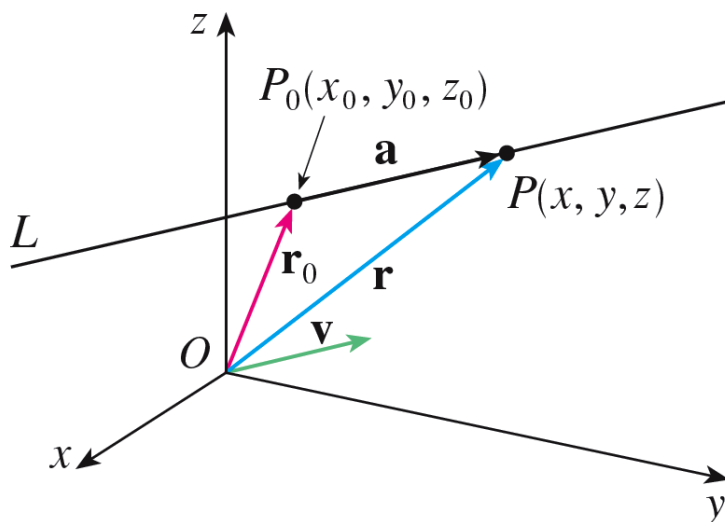


Lines and Planes in Space

In the **plane**, a **line** is determined by a **point** and a **number giving the slope** of the line. In **space** a **line** is determined by a **point** and a **vector** giving the direction of the line.

Equations for a line

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $v = v_1i + v_2j + v_3k$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to v .



The standard equation of the line through $P_0(x_0, y_0, z_0)$ **parallel** to $v = v_1i + v_2j + v_3k$ is:

$$x = x_0 + tv, \quad y = y_0 + tv, \quad z = z_0 + tv, \quad -\infty < t < \infty$$

$$\text{and } (x, y, z) = (x_0 + tv, y_0 + tv, z_0 + tv)$$

Note: To write the equation of a line in 3D space, we need a point on the line and a parallel vector to the line.

Example: Find the equations for the line through $(-2, 0, 4)$ parallel to $v = 2i + 4j - 2k$.

Solution

With $P_0(x_0, y_0, z_0)$ equal to $(-2, 0, 4)$ and $v = v_1i + v_2j + v_3k$ equal to $v = 2i + 4j - 2k$

$$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t$$



Example: Find the equations for the line through $P(-3,2,-3)$ and $Q(1,-1,4)$.

Solution: The vector $\overrightarrow{PQ} = 4i - 3j + 7k$ is parallel to the line and equation with

$(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t \quad , \quad y = 2 - 3t \quad , \quad z = -3 + 7t$$

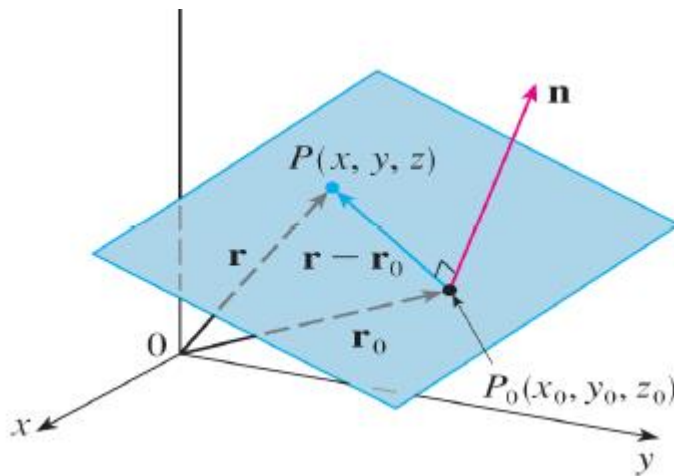
We could have choose $Q(1,-1,4)$

$$x = 1 + 4t \quad , \quad y = -1 - 3t \quad , \quad z = 4 + 7t$$

Equation of the Plane in space

Consider the plane containing the point $P_0 = (x_0, y_0, z_0)$ and normal vector $\mathbf{n} = [a, b, c]$ perpendicular to the plane.

$$\mathbf{n} = [A, B, C]$$



The plane consists of all points $P = (x, y, z)$ for which the vector $\overrightarrow{PP_0}$ is orthogonal to the normal vector $\mathbf{n} = [A, B, C]$. Since $\overrightarrow{PP_0}$ and $\mathbf{n}$ are orthogonal, the following equations hold:

$$\mathbf{n} \cdot \overrightarrow{PP_0} = 0$$

$$[A, B, C] \cdot [x - x_0, y - y_0, z - z_0] = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$



This gives the *standard equation of a plane*. If we expand this equation we obtain the following equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$\text{or } Ax + By + Cz = D \quad , \quad \text{where } D = Ax_0 + By_0 + Cz_0$$

Example.: Find an equation for the plane through $P_0(-3,0,7)$ perpendicular to

$$n = 5i + 2j - k .$$

Solution

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0$$

$$5x + 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22$$

Notice in this example how the components of $n = 5i + 2j - k$ become the coefficients of x , y and z in equation $5x + 2y - z = -22$. The vector $n = Ai + Bj + Ck$ is normal to the plane $Ax + By + Cz = D$.

Ex.: Find an equation for the plane through $A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$.

Solution

We find a vector **normal** to the plane and use it with one of the point to write an equation for the plane.

The cross product:

$$AB \times AC = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3i + 2j + 6k \text{ is } \mathbf{normal} \text{ to the plane.}$$

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$

$$3x + 2y + 6z = 6$$



Lines of intersection

- Two lines are parallel if and only if they have the same direction.
- Two planes are parallel if and only if their normals are parallel.
- The planes that are not parallel **intersect in a line**.

Ex.: Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution: The line of intersection of two planes is perpendicular to both **planes'** **normal** vectors n_1 and n_2 and therefore parallel to $n_1 \times n_2$. i.e. $n_1 \times n_2$ is a vector

parallel to the planes' line of intersection.
$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14i + 2j + 15k$$

Ex.: Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$

Solution

The point $\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8$$

$$t = -1$$

The point of intersection is $(x, y, z) \Big|_{t=-1} = \left(\frac{2}{3}, 2, 0\right)$

Angles between planes

The angle between two intersecting planes is defined to be the angle determined by **their normal vectors**.

Ex.: Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$



Solution

The vectors $n_1 = 3i - 6j - 2k$ and $n_2 = 2i + j - 2k$ are normal to the planes. The angle between them is

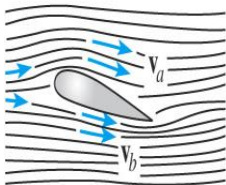
$$\theta = \cos^{-1} \left(\frac{n_1 \cdot n_2}{|n_1| |n_2|} \right)$$

$$= \cos^{-1} \left(\frac{4}{21} \right)$$

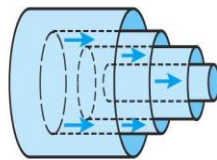
Div, Grad, Curl & the Laplacian

Div, Grad and Curl are used to give information about a vector field. (**A vector field is a field where each point can be described as a vector.**) All three of these require knowledge of calculus, namely differentiation and partial differentiation. Div, grad and curl all utilise the symbol ∇ (del or nabla).

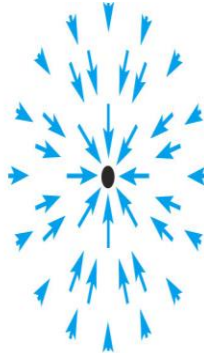
∇ represents the vector $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$



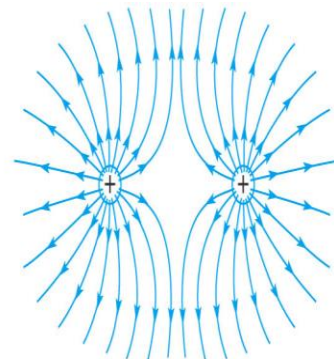
(a) Airflow around an airplane wing



(b) Laminar flow of blood in an artery; cylindrical layers of blood flow faster near the center of the artery



(c) Inverse square force field; magnitude of the attractive force is large near the particle



(d) Lines of force around two equal positive charges

Divergence

Div is used to measure the density of a vector field in order to see how quickly matter is moving into and out of a region.

Div is calculated from a vector and the result is a scalar.



$\nabla \cdot F$ is used to denote divergence.

Divergence is found by taking the dot product of ∇ and the vector.

$$\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Here F_x, F_y, F_z are the components of the vector F .

To calculate Div differentiate the first component with respect to x , then differentiate the second component with respect to y and differentiate the third component with respect to z . Finally, add the results of the 3 differentiations together.

Examples:

1 If $F = 2x\hat{i} + 3xy\hat{j} + z^2\hat{k}$, then

$$\begin{aligned}\nabla \cdot F &= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3xy) + \frac{\partial}{\partial z}(z^2) \\ &= 2 + 3x + 2z\end{aligned}$$

2 If $F = x^2\hat{i} + 2z\hat{j} - 3xy\hat{k}$ then

$$\begin{aligned}\nabla \cdot F &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(-3xy) \\ &= 2x + 0 + 0 = 2x\end{aligned}$$

Note that the results do not include $\hat{i}, \hat{j}$, or $\hat{k}$, they are scalars.

Exercise: Find the divergence of the following vector fields:

- a) $F = 4x^2y\hat{i} + 2yz\hat{j} - 3z\hat{k}$
- b) $F = 3y\hat{i} + 2x^2y\hat{j} + 2yz\hat{k}$
- c) $F = 3\cos 2x\hat{i} + 3y\sin 2x\hat{j} + 2z\cos x\hat{k}$
- d) $F = 3e^x\hat{i} + y^2e^x\hat{j} + e^{2xz}\hat{k}$

Gradient: The gradient or Grad is used to find where in the field the gradient is changing the most rapidly. The gradient function is used on a scalar and the result is a vector.

$$\nabla f = \frac{\partial}{\partial x}(f)\hat{i} + \frac{\partial}{\partial y}(f)\hat{j} + \frac{\partial}{\partial z}(f)\hat{k}$$



Here f is the scalar that you are differentiating.

To calculate Grad differentiate the whole function with respect to x . This gives the $\hat{i}$ component. Next differentiate the original function with respect to y . This gives the $\hat{j}$ component. Finally repeat for z , giving the $\hat{k}$ component.

Examples

1 If $f(x, y, z) = 2z^2 + 3yx$ then

$$\begin{aligned}\nabla f &= \frac{\partial}{\partial x}(2z^2 + 3yx)\hat{i} + \frac{\partial}{\partial y}(2z^2 + 3yx)\hat{j} + \frac{\partial}{\partial z}(2z^2 + 3yx)\hat{k} \\ &= 3y\hat{i} + 3x\hat{j} + 4z\hat{k}\end{aligned}$$

2 If $f(x, y, z) = 5xyz^2 - 2xy + \cos x$ then

$$\begin{aligned}\nabla f &= \frac{\partial}{\partial x}(5xyz^2 - 2xy + \cos x)\hat{i} + \frac{\partial}{\partial y}(5xyz^2 - 2xy + \cos x)\hat{j} \\ &\quad + \frac{\partial}{\partial z}(5xyz^2 - 2xy + \cos x)\hat{k} \\ &= (5yz^2 - 2y - \sin x)\hat{i} + (5xz^2 - 2x)\hat{j} + (10xyz)\hat{k}\end{aligned}$$

Exercise : Find the gradient vector for the following:

- a) $f = x^2y + 3yz + 2xz^2$
- b) $f = 3x^2y^2z^3$
- c) $f = x\cos y + z^2\sin x + xy\sin z$

Curl

Curl is used to determine the amount of rotation of a vector field. If the curl of a vector field is zero, then it is said to be 'irrotational', i.e. it does not rotate. The curl is used on a vector and the result of calculating the curl is also a vector.

Calculating the curl is very similar to calculating the cross product of two vectors and is denoted $\nabla \times F$. The main difference is that the first vector that is used here is

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

The formula for curl is:



$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}$$

To calculate the curl, first write out the determinant with $\hat{i}, \hat{j}, \hat{k}$ as the top row, ∇ as the second row and the vector F as the third. The next step is to work it out as if it were the cross product. The difference here is that instead of finding pairs to multiply together, you are finding out which part of F will be differentiated by x, y , or z

Examples

1 If $F = x^2 \hat{i} + 2z \hat{j} - 3xy \hat{k}$ then

$$\begin{aligned} \nabla \times F &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2z & -3xy \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(-3xy) - \frac{\partial}{\partial z}(2z) \right) \hat{i} - \left(\frac{\partial}{\partial x}(-3xy) - \frac{\partial}{\partial z}(x^2) \right) \hat{j} + \left(\frac{\partial}{\partial x}(2z) - \frac{\partial}{\partial y}(x^2) \right) \hat{k} \\ &= (-3x - 2) \hat{i} - (-3y - 0) \hat{j} + (0 - 0) \hat{k} \\ &= (-3x - 2) \hat{i} + 3y \hat{j} \end{aligned}$$

2 If $F = 2xy^2z \hat{i} + 3xy \hat{j} + z^2 \hat{k}$, then

$$\begin{aligned} \nabla \times F &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2z & 3xy & z^2 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(3xy) \right) \hat{i} - \left(\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(2xy^2z) \right) \hat{j} + \left(\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(2xy^2z) \right) \hat{k} \\ &= (0 - 0) \hat{i} - (0 - 2xy^2) \hat{j} + (3y - 4xyz) \hat{k} \\ &= 2xy^2 \hat{j} + (3y - 4xyz) \hat{k} \end{aligned}$$

Exercise:

Calculate the curl of the following vector fields:

a) $F = x^2y \hat{i} + z^3y \hat{j} + x \hat{k}$

b) $F = y^2z \hat{i} + x^3z^2 \hat{j} + 3y \hat{k}$

c) $F = 3y \cos 2x \hat{i} + 3x \sin 2z \hat{j} + \cos y \hat{k}$



$\text{grad}(\text{scalar function}) = \text{Vector Field}$

$\text{div}(\text{Vector Field}) = \text{scalar function}$

$\text{curl}(\text{Vector Field}) = \text{Vector Field}$

Possible combinations of Div, Grad and Curl

Which combinations work depends on whether the result of the first process produces a vector or a scalar.

As Grad produces a vector, it is possible to find Div(Grad) and Curl(Grad). As Div produces a scalar it is possible to find Grad(Div). As Curl produces a vector, it is possible to find Div(Grad), and as Curl also works on vectors, you can find Curl(Curl) too.

Which of the 9 ways to combine grad, div and curl by taking one of each. Which of these combinations make sense?

~~$\text{grad}(\text{grad}(f))$~~ Vector Field	$\checkmark \text{div}(\text{grad}(f))$ Vector Field	$\checkmark \text{curl}(\text{grad}(f))$ Vector Field 0 vector
$\checkmark \text{grad}(\text{div}(\mathbf{F}))$ scalar function	~~$\text{div}(\text{div}(\mathbf{F}))$~~ scalar function	~~$\text{curl}(\text{div}(\mathbf{F}))$~~ scalar function
~~$\text{grad}(\text{curl}(\mathbf{F}))$~~ Vector Field	$\checkmark \text{div}(\text{curl}(\mathbf{F}))$ Vector Field 0 scalar	$\checkmark \text{curl}(\text{curl}(\mathbf{F}))$ Vector Field

The Laplacian

The Laplacian is denoted by ∇^2 .

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The Laplacian works on scalars and produces a scalar result.

To find $\nabla^2 f$, find the first derivative of f with respect to x , y , and z . Then find the second derivative of each and, finally add up your results



Example: Given $f = 3x^2y + xz^2$, find $\nabla^2 f$.

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\&= \frac{\partial^2}{\partial x^2}(3x^2y + xz^2) + \frac{\partial^2}{\partial y^2}(3x^2y + xz^2) + \frac{\partial^2}{\partial z^2}(3x^2y + xz^2) \\&= \frac{\partial}{\partial x}(6xy + z^2) + \frac{\partial}{\partial y}(3x^2) + \frac{\partial}{\partial z}(2xz) \\&= 6y + 0 + 2x = 6y + 2x\end{aligned}$$

Exercise:

Find the Laplacian of the following functions:

- a) $x^2 + 3yz^3$
- b) $xy^2z^4 + 3x^3y$