

Ex ① Find r, θ if $z_1 = -\sqrt{5}i, z_2 = -\frac{3}{2}i + \frac{3}{2}$

Solution

$$|z_1| = r_1 = \sqrt{(-\sqrt{5})^2 + 0^2} = \sqrt{5} = r_1$$

$$\theta_1 = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{-\sqrt{5}}{0} = (\text{undefined})$$

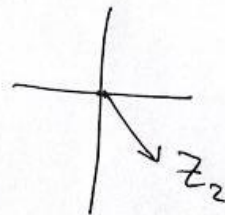
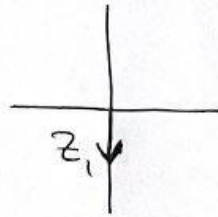
$$\Rightarrow \theta_1 = \left(\frac{\pi}{2} \text{ or } \frac{3\pi}{2}\right) \Rightarrow \text{y-axis } z_1 \Rightarrow \theta_1 = 270^\circ = \frac{3\pi}{2}$$

$$r_2 = |z_2| = \sqrt{\left(+\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{9}{4}} = \sqrt{\frac{18}{4}} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} = r_2$$

$$z_2 = \frac{3}{2} - \frac{3}{2}i$$

$$\theta_2 = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{-\frac{3}{2}}{\frac{3}{2}} = \tan^{-1}(-1) = -\frac{\pi}{4}$$

$$\theta = 2\pi - \frac{\pi}{4} = \frac{8\pi - \pi}{4} = \frac{7\pi}{4} = \theta_2$$



② Let $z_1 = 1+i, z_2 = \sqrt{3} - i$ find $(z_1 + z_2)$ and (z_1/z_2) in polar

Solution

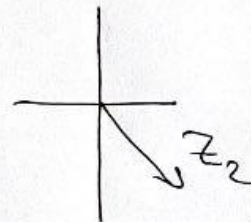
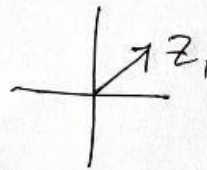
$$r_1 = |z_1| = \sqrt{1^2 + 1^2} = \sqrt{2} = r_1$$

$$\theta_1 = \tan^{-1} \frac{1}{1} = \boxed{\frac{\pi}{4}} = \theta_1$$

$$r_2 = |z_2| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \boxed{2} = r_2$$

$$\theta_2 = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{-1}{\sqrt{3}} = -30^\circ = -\frac{\pi}{6} = \theta_2$$

$$\theta_2 = 2\pi - \frac{\pi}{6} = \frac{12\pi - \pi}{6} = \boxed{\frac{11\pi}{6}} = \theta_2$$



$$\text{in polar } (Z_1 Z_2) = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$Z_1 Z_2 = 2\sqrt{2} [\cos(\frac{\pi}{4} + \frac{11\pi}{6}) + i \sin(\frac{\pi}{4} + \frac{11\pi}{6})]$$

$$Z_1 Z_2 = 2\sqrt{2} [\cos \frac{25\pi}{12} + i \sin \frac{25\pi}{12}]$$

$$(Z_1 / Z_2) = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

$$= \frac{2}{2} [\cos(\frac{\pi}{4} - \frac{11\pi}{6}) + i \sin(\frac{\pi}{4} - \frac{11\pi}{6})]$$

$$= \frac{\sqrt{2}}{2} [\cos(-\frac{19\pi}{12}) + i \sin(-\frac{19\pi}{12})]$$

$$Z_1 / Z_2 = \frac{\sqrt{2}}{2} [\cos(\frac{19\pi}{12}) - i \sin(\frac{19\pi}{12})]$$

(3) Find $(2+2i)^5$ in the form of $a+ib$ using polar form

Solution

$$Z = 2+2i \Rightarrow r = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} = r$$

$$\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{2}{2} = \tan^{-1} 1 = \boxed{\frac{\pi}{4}} \text{ (acute angle)}$$

$$\therefore Z^n = r^n [\cos n\theta + i \sin n\theta] \quad n=5$$

$$\therefore Z^5 = (2+2i)^5 = (2\sqrt{2})^5 [\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}] \quad \left[\frac{5\pi}{4} = 225^\circ \right]$$

$$= (2^5) (\sqrt{2})^5 \left[-\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right]$$

$$= (32) (4\sqrt{2}) \left[-\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right]$$

$$= 128\sqrt{2} \left[-\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right] = \underline{\underline{-128 - 128i}}$$

(9)

De Moivre's Formula :

To find the roots $z^{\frac{1}{n}}$ of z we use De Moivre's formula as follows: $\{z^{\frac{1}{n}} \text{ has } \underline{n} \text{ complex roots}\}$

$$C^{\frac{1}{n}} = \sqrt[n]{z} = z^{\frac{1}{n}} = \sqrt[n]{r} e^{i\left(\frac{\theta + 2k\pi}{n}\right)} \quad k = 0, 1, 2, \dots, n-1$$

$$= r^{\frac{1}{n}} \left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right]$$

Ex Find the roots of $(-2\sqrt{3} - 2i)^{\frac{1}{4}}$

Solu $z = -2\sqrt{3} - 2i \quad r = 4, \theta = \frac{\pi}{6}$

$$\sqrt[4]{z} = 4^{\frac{1}{4}} \left[\cos \frac{\frac{\pi}{6} + 2k\pi}{4} + i \sin \frac{\frac{\pi}{6} + 2k\pi}{4} \right] \quad k = 0, 1, 2, 3$$

when

$$k=0 \Rightarrow z_1 = 4^{\frac{1}{4}} \left[\cos \frac{\pi}{24} + i \sin \frac{\pi}{24} \right]$$

$$k=1 \Rightarrow z_2 = 4^{\frac{1}{4}} \left[\cos \frac{13\pi}{24} + i \sin \frac{13\pi}{24} \right]$$

$$k=2 \Rightarrow z_3 = 4^{\frac{1}{4}} \left[\cos \frac{25\pi}{24} + i \sin \frac{25\pi}{24} \right]$$

$$k=3 \Rightarrow z_4 = 4^{\frac{1}{4}} \left[\cos \frac{37\pi}{24} + i \sin \frac{37\pi}{24} \right]$$

Ex (17) Find the roots of $\sqrt[2]{i}$

(2) Solve the equation $z^3 + 1 = 0$

(5)

Ex Find the roots of $(-2\sqrt{3} - 2i)^{\frac{1}{4}}$

Solution

$$Z = -2\sqrt{3} - 2i$$

$$r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{4 \cdot 3 + 4} = \sqrt{16} = \underline{4 = r}$$

$$\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{-2}{-2\sqrt{3}} = \tan^{-1} \frac{1}{\sqrt{3}} = 30^\circ = \frac{\pi}{6} \quad (\text{2nd/3rd Q})$$

$$\therefore \theta = \pi + \frac{\pi}{6} = \boxed{\frac{7\pi}{6}} = \theta$$

$$\sqrt[n]{Z} = r^{\frac{1}{n}} \left[\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right]$$

$$n = 4, \quad k = 0, 1, 2, 3$$

When $k=0$:

$$\begin{aligned} & 4^{\frac{1}{4}} \left[\cos \frac{\frac{7\pi}{6} + 2(0)\pi}{4} + i \sin \frac{\frac{7\pi}{6} + 2(0)\pi}{4} \right] \\ &= 4^{\frac{1}{4}} \left[\cos \frac{7\pi}{24} + i \sin \frac{7\pi}{24} \right] \end{aligned}$$

$k=1$

$$\begin{aligned} &= 4^{\frac{1}{4}} \left[\cos \frac{\frac{7\pi}{6} + 2(1)\pi}{4} + i \sin \frac{\frac{7\pi}{6} + 2(1)\pi}{4} \right] \\ &= 4^{\frac{1}{4}} \left[\cos \frac{19\pi}{24} + i \sin \frac{19\pi}{24} \right] \end{aligned}$$

$k=2$

$$\begin{aligned} & 4^{\frac{1}{4}} \left[\cos \frac{\frac{7\pi}{6} + 2(2)\pi}{4} + i \sin \frac{\frac{7\pi}{6} + 2(2)\pi}{4} \right] \\ &= 4^{\frac{1}{4}} \left[\cos \frac{31\pi}{24} + i \sin \frac{31\pi}{24} \right] \end{aligned}$$

$k=3$

$$\begin{aligned} & 4^{\frac{1}{4}} \left[\cos \frac{\frac{7\pi}{6} + 2(3)\pi}{4} + i \sin \frac{\frac{7\pi}{6} + 2(3)\pi}{4} \right] \\ &= 4^{\frac{1}{4}} \left[\cos \frac{43\pi}{24} + i \sin \frac{43\pi}{24} \right] \end{aligned}$$

Ex Find the Complex roots $(1 - \sqrt{3}i)^{\frac{1}{3}}$

Solution Using De Moivre's method

$$Z = 1 - \sqrt{3}i \Rightarrow r = \sqrt{1 + (-\sqrt{3})^2} = 2 = r$$

$$\theta = \tan^{-1} \frac{-\sqrt{3}}{1} = \tan^{-1} -\sqrt{3} \Rightarrow \theta = -60^\circ = -\frac{\pi}{3}$$

$$\therefore \theta = 2\pi - \frac{\pi}{3} = \frac{6\pi - \pi}{3} = \boxed{\frac{5\pi}{3} = \theta}$$

$$n=3, k=0, 1, 2$$

$$\therefore Z^{\frac{1}{n}}: Z^{\frac{1}{3}} = r^{\frac{1}{n}} \left[\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right]$$

$$k=0 \quad Z_0 = 2^{\frac{1}{3}} \left[\cos \frac{\frac{5\pi}{3} + 2(0)\pi}{3} + i \sin \frac{\frac{5\pi}{3} + 2(0)\pi}{3} \right]$$

$$= 2^{\frac{1}{3}} \left[\cos \frac{5\pi}{9} + i \sin \frac{5\pi}{9} \right]$$

$$k=1 \Rightarrow Z_1 = 2^{\frac{1}{3}} \left[\cos \frac{\frac{5\pi}{3} + 2(1)\pi}{3} + i \sin \frac{\frac{5\pi}{3} + 2(1)\pi}{3} \right]$$

$$= 2^{\frac{1}{3}} \left[\cos \frac{11\pi}{9} + i \sin \frac{11\pi}{9} \right]$$

$$k=2 \Rightarrow Z_2 = 2^{\frac{1}{3}} \left[\cos \frac{17\pi}{9} + i \sin \frac{17\pi}{9} \right]$$

Ex $Z = (1-i)^{\frac{1}{3}} (2i)^{\frac{1}{3}}$

Solution $Z = (2i + 2)^{\frac{1}{3}} = (2 + 2i)^{\frac{1}{3}} \quad n=3, k=0, 1, 2$

$$r = \sqrt{u^2 + v^2} = \sqrt{8} = 2\sqrt{2}, \quad \theta = \tan^{-1} \frac{2}{2} = \tan^{-1} 1 = \theta = \frac{\pi}{4}$$