

Notation of the Polar Form

If $Z = x + iy$ $x, y \in \mathbb{R}$ in polar we write
 $x = r \cos \theta$, $y = r \sin \theta$ where $r = \sqrt{x^2 + y^2}$

$$\therefore Z = r(\cos \theta + i \sin \theta)$$

it is called trigonometric form, Sometimes we write

$$\text{cis } \theta = \cos \theta + i \sin \theta \Rightarrow Z = r \text{cis } \theta \text{ or } Z = r e^{i\theta}$$

which is called exponential form.

Operations on Complex numbers in Polar Form

$$\text{let } Z_1 = r_1(\cos \theta_1 + i \sin \theta_1), Z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

$$(1) \text{ Multiplication: } Z_1 Z_2 = r_1 r_2 [\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)]$$

$$\boxed{Z_1 Z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]}$$
$$= r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\text{and } |Z_1 Z_2| = r_1 r_2, \arg Z_1 Z_2 = \theta_1 + \theta_2$$

$$(2) \text{ Division: } \boxed{Z_1 / Z_2 = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]}$$
$$= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

(7)

$$\text{and } \left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2}, \quad \arg \frac{z_1}{z_2} = \theta_1 - \theta_2$$

$$(3) \text{ Powers: } z^n = r^n [\cos n\theta + i \sin n\theta]$$

$$z^n = r^n e^{in\theta}$$

Ex If $z_1 = -5+5i$, $z_2 = -3i$

Find $z_1 z_2$ and z_1/z_2 using both Cartesian and Polar Form:-

Solution:- in Cartesian: $z_1 z_2 = (-5+5i)(-3i) = 15+15i$

in polar:- $z_1 = -5+5i \Rightarrow |z_1| = \sqrt{25+25} = \sqrt{50} = \underline{5\sqrt{2}} = r_1$

$z_2 = -3i \Rightarrow |z_2| = \sqrt{9} = \underline{3} = r_2$

$\theta_1 = \tan^{-1} \frac{y_1}{x_1} \Rightarrow \theta_1 = \tan^{-1} \frac{5}{-5} \Rightarrow \theta_1 = \tan^{-1} -1 \Rightarrow \theta_1 = -\frac{\pi}{4}$ or $\frac{3\pi}{4}$

$\theta_2 = \tan^{-1} \frac{y_2}{x_2} \Rightarrow \theta_2 = \tan^{-1} \frac{-3}{0}$ (undefined) $\Rightarrow \theta_2 = \frac{3\pi}{2}$

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \\ &= 15\sqrt{2} [\cos(3\frac{\pi}{4} + 3\frac{\pi}{2}) + i \sin(3\frac{\pi}{4} + 3\frac{\pi}{2})] \\ &= 15\sqrt{2} [\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4}] \end{aligned} \quad (4)$$

$$\begin{aligned} z_1/z_2 \text{ in polar form is } & \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \\ &= \frac{5\sqrt{2}}{3} [\cos(3\frac{\pi}{4} - 3\frac{\pi}{2}) + i \sin(3\frac{\pi}{4} - 3\frac{\pi}{2})] \\ &= \frac{5\sqrt{2}}{3} [\cos(-3\frac{\pi}{4}) + i \sin(-3\frac{\pi}{4})] \\ &= \underline{z_1/z_2} = \frac{5\sqrt{2}}{3} [\cos(3\frac{\pi}{4}) - i \sin(3\frac{\pi}{4})] \end{aligned}$$

Ex If $z_1 = -5 + 5i$, $z_2 = -3i$

Find $z_1 z_2$ and (z_1 / z_2) using both Cartesian and polar form

Solution: In Cartesian

$$z_1 z_2 = (-5 + 5i)(-3i) = 15i + 15 = 15 + 15i$$

$$\frac{z_1}{z_2} = \frac{-5 + 5i}{-3i} \cdot \frac{i}{i} = \frac{-5i - 5}{3} = -\frac{5}{3} - \frac{5}{3}i$$

In polar!

$$z_1 = -5 + 5i \Rightarrow |z_1| = r_1 = \sqrt{(-5)^2 + 5^2} = \sqrt{50} = \boxed{5\sqrt{2} = r_1}$$

$$z_2 = -3i \Rightarrow |z_2| = r_2 = \sqrt{(0)^2 + (-3)^2} = \sqrt{9} = \boxed{3 = r_2}$$

$$\theta_1 = \tan^{-1} \frac{y_1}{x_1} \Rightarrow \theta_1 = \tan^{-1} \frac{5}{-5} \Rightarrow \theta_1 = \tan^{-1}(-1) = -\frac{\pi}{4} \text{ (Clockwise)}$$

$$\Rightarrow \theta_1 = (\pi - \frac{\pi}{4}) = \frac{3\pi}{4}$$

$$\theta_2 = \tan^{-1} \frac{y_2}{x_2} = \tan^{-1} \frac{-3}{0} \text{ (undefined)} \Rightarrow \boxed{\theta_2 = \frac{3\pi}{2}}$$

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \\ &= (5\sqrt{2})(3) [\cos(\frac{3\pi}{4} + \frac{3\pi}{2}) + i \sin(\frac{3\pi}{4} + \frac{3\pi}{2})] \\ &= 15\sqrt{2} [\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4}] \quad (\frac{9\pi}{4} = 405^\circ) \\ z_1 z_2 &= 15\sqrt{2} [\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}] \end{aligned}$$

$$\begin{aligned} z_1 / z_2 &= \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \\ &= \frac{5\sqrt{2}}{3} [\cos(\frac{3\pi}{4} - \frac{3\pi}{2}) + i \sin(\frac{3\pi}{4} - \frac{3\pi}{2})] \quad (\frac{3\pi}{4} = 135^\circ) \\ &= \frac{5\sqrt{2}}{3} [\cos(-\frac{3\pi}{4}) + i \sin(-\frac{3\pi}{4})] \\ &= \frac{5}{3} \sqrt{2} [\cos(\frac{3\pi}{4}) - i \sin(\frac{3\pi}{4})] = \frac{5}{3} \sqrt{2} [-\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}] \end{aligned}$$

Example use the polar form to put the complex number $(-\sqrt{3} + i)^3$ in the $a+ib$ form.

Solution $z^n = r^n [\cos(n\theta) + i \sin(n\theta)]$
 $z^n = r^n e^{in\theta}$

$$z = -\sqrt{3} + i \Rightarrow r = |z| = \sqrt{(-\sqrt{3})^2 + 1} = \sqrt{3+1} = \underline{2 = r}$$

$$\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{1}{-\sqrt{3}} = -30^\circ = -\frac{\pi}{6} \quad (\text{مربع اول، ربع ثانی})$$

$$\therefore \theta = \pi - \frac{\pi}{6} = \underline{\frac{5\pi}{6} = \theta}$$

$$z^3 = (-\sqrt{3} + i)^3 = 2^3 e^{i3 \cdot \frac{5\pi}{6}} = 2^3 e^{i \frac{5\pi}{2}}$$

$$= 2^3 \left[\cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2} \right]$$

$$= 8 [0 + i(1)] = 8i = 0 + 8i$$

$a+ib$

H.W

(1) let $z_1 = 1+i$, $z_2 = \sqrt{3} - i$ find $z_1 \cdot z_2$
 z_1/z_2 in polar.

(2) Find $(2+2i)^5$ in the form of $a+ib$
using Polar form.

H.w) let $z_1 = 1+i$, $z_2 = \sqrt{3} - i$ find $z_1 z_2$, z_1/z_2 in polar form.

Ex Solve for the real numbers x and y

(1) $(3+4i)^2 - 2(x-iy) = x+iy$

Solu! $9 + 24i - 16 - 2x + 2iy = x + iy$

$(-7 - 2x) + i(24 + 2y) = x + iy$

$-7 - 2x = x \Rightarrow -7 = 3x \Rightarrow \boxed{x = -\frac{7}{3}}$

$24 + 2y = y \Rightarrow 24 = -y \Rightarrow \boxed{y = -24}$

(2) $\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+iy} = 1+i$

$\frac{1+2i-1}{1-2i-1} + \frac{1}{x+iy} = 1+i \Rightarrow \frac{2i}{-2i} + \frac{1}{x+iy} = 1+i$

$-1 + \frac{1}{x+iy} = 1+i \Rightarrow \frac{1}{x+iy} = 2+i \Rightarrow (2+i)(x+iy) = 1$

$(2x-y) + i(x+2y) = 1+i \Rightarrow \begin{cases} 2x-y=1 \\ x+2y=0 \end{cases} \Rightarrow$

$y = -\frac{1}{5}, x = \frac{2}{5}$

H.w) $(3-2i)(x+iy) = 2(x-2iy) + 2i - 1$