

Lecture (1)

Complex Number System

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The Complex Number System

Background:

1. Let a and b be real numbers with $a \neq 0$.
There is a real number r that satisfies the
equation

$$ax + b = 0;$$

The equation $ax + b = 0$ is a *linear equation*
in one variable.

2. Let a , b , and c be real numbers with $a \neq 0$. Does there exist a real number r which satisfies the equation

$$ax^2 + bx + c = 0?$$

Answer: Not necessarily; sometimes “yes”, sometimes “no”.

The equation $ax^2 + bx + c = 0$
is a *quadratic equation in one variable*.

Examples:

1. $x^2 - 5x + 6 = 0$; roots : $r_1 = 2, r_2 = 3$.

2. $x^2 - 2x + 5 = 0$; no real roots!

3. Simple case:

$$x^2 + 1 = 0; \quad \text{no real roots}$$

The imaginary number i

DEFINITION: The imaginary number i is a root of the equation

$$x^2 + 1 = 0.$$

($-i$ is also a root of this equation.)

ALTERNATE DEFINITION: $i^2 = -1$ or

$$i = \sqrt{-1}.$$

The Complex Number System

DEFINITION: The set C of *complex numbers* is given by

$$C = \{a + bi \mid a, b \in R\}.$$

NOTE: The set of real numbers is a subset of the set of complex numbers; $R \subset C$, since

$$a = a + 0i \quad \text{for every } a \in R.$$

Some terminology

Given the complex number $z = a + bi$.

- The real number a is called the *real part* of z .

- The real number b is called the *imaginary part* of z .

- The complex number $\bar{z} = a - bi$ is called the *conjugate* of z .

Arithmetic of Complex Numbers

Let a , b , c , and d be real numbers.

Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Subtraction: $(a + bi) - (c + di) = (a - c) + (b - d)i$

Multiplication: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Division:

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i\end{aligned}$$

provided $c^2 + d^2 \neq 0$

Field Axioms

The set of complex numbers $\mathbb{C}$ satisfies the field axioms:

- **Addition** is commutative and associative, $0 = 0 + 0i$ is the additive identity, $-a - bi$ is the additive inverse of $a + bi$.

- **Multiplication** is commutative and associative, $1 = 1 + 0i$ is the multiplicative identity, $\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$ is the multiplicative inverse of $a + bi$.

and

- the **Distributive Law** holds. That is,

if α , β , and γ are complex numbers, then

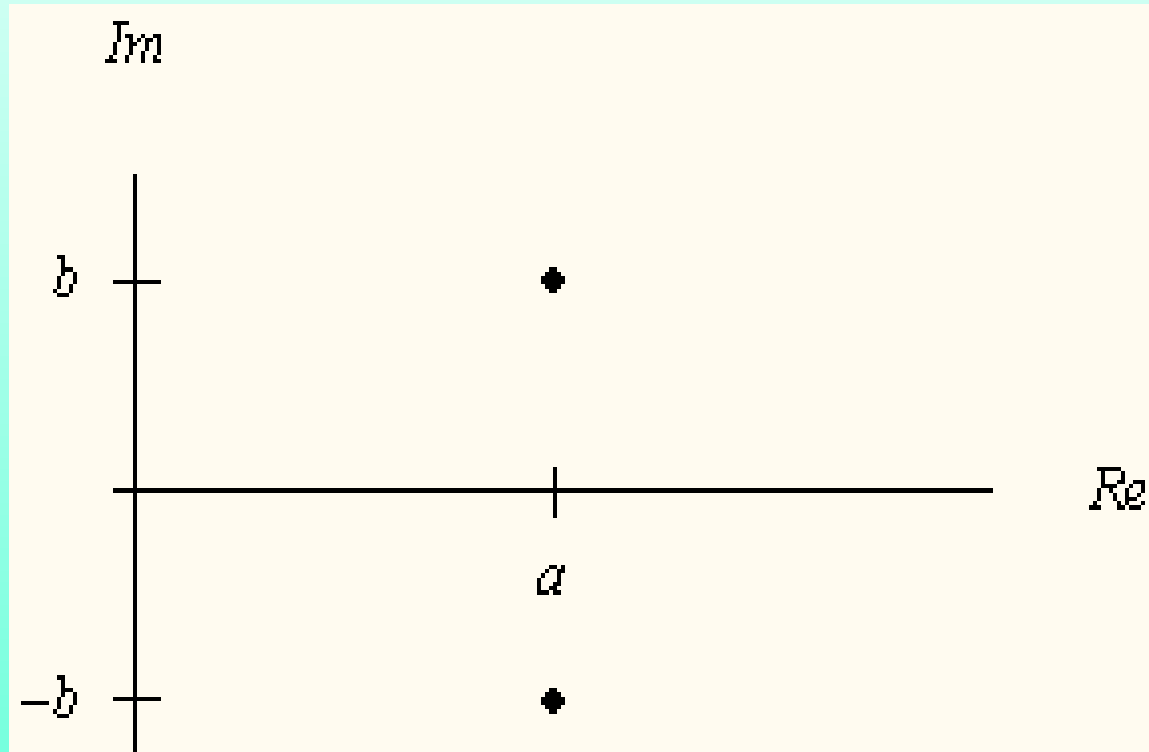
$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$$

“Geometry” of the Complex Number System

A complex number is a number of the form $a + bi$, where a and b are real numbers.

If we “identify” $a + bi$ with the ordered pair of real numbers (a,b) we get a point in a coordinate plane – which we call the *complex plane*.

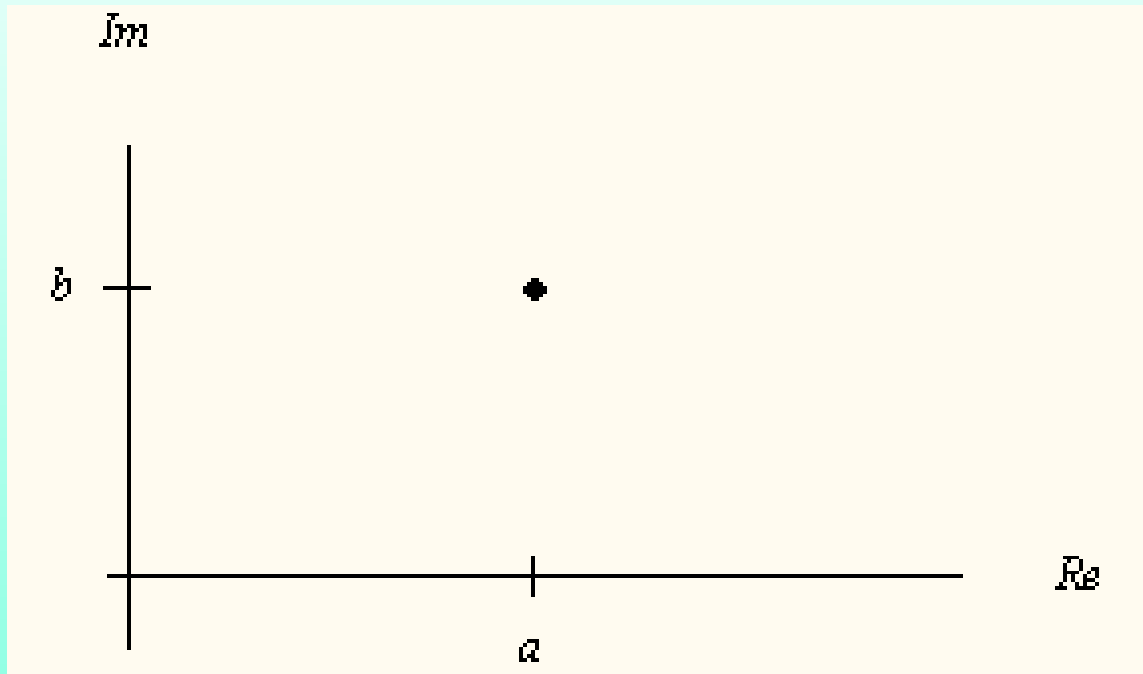
The Complex Plane



Absolute Value of a Complex Number

Recall that the absolute value of a real number a is the distance from the point a (on the real line) to the origin 0 .

The same definition is used for complex numbers.



$$|a + bi| = \sqrt{a^2 + b^2}$$

Fundamental Theorem of Algebra

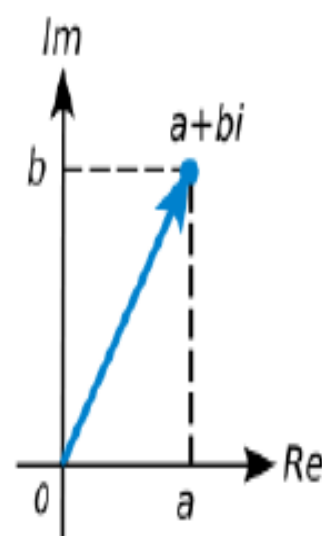
A polynomial of degree $n \geq 1$

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

has exactly n (complex) roots.

Complex Numbers in Linear Algebra

A complex number is a number consisting of a real part and an imaginary part. Complex numbers extend the idea of the one-dimensional number line to the two-dimensional complex plane by using the number line for the real part and adding a vertical axis to plot the imaginary part. In this way the complex numbers contain the ordinary real numbers while extending them in order to solve problems that would be impossible with only real numbers.



Introduction and definition

Complex numbers have been introduced to allow for solutions of certain equations that have no real solution:

the equation $x^2+1=0$

has no real solution x , since the square of x is 0 or positive, so $x^2 + 1$ cannot be zero. Complex numbers are a solution to this problem. The idea is to enhance the real numbers by adding a number i whose square is -1 , so that $x = i$ and $x = -i$ are the two solutions to the preceding equation.

[edit]Definition

A complex number is an expression of the form

where a and b are real numbers and i is a mathematical symbol which is called the imaginary unit. For example, $-3.5 + 2i$ is a complex number.

Elementary operations

Addition and subtraction

Addition of two complex numbers can be done geometrically by constructing a parallelogram.

Complex numbers are added by adding the real and imaginary parts of the summands. That is to say:

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

Similarly, subtraction is defined by

$$(a+bi)-(c+di) = (a-c)+(b-d)i$$

Multiplication and division

The multiplication of two complex numbers is defined by the following formula:

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

In particular, the square of the imaginary unit is -1 :

$$i^2 = \ddot{i} = -1$$

The division of two complex numbers is defined by the following formula:

$$(a+bi)/(c+di) = (ac+bd)/(c^2+d^2) + i(bc-ad)/(c^2+d^2)$$

Complex Matrix

Definition: A matrix whose elements may contain complex numbers.

$$\text{Let } A = \begin{pmatrix} 4+i & -2+3i \\ 6+4i & -3i \end{pmatrix} \quad B = \begin{pmatrix} 2-i & 3-4i \\ 5+2i & -7+5i \end{pmatrix} \quad C = \begin{pmatrix} 1+2i & i \\ 3-i & 8 \\ 4+2i & 1-i \end{pmatrix}$$

$$(a) \quad A+B = \begin{pmatrix} (4+i)+(2-i) & (-2+3i)+(3-4i) \\ (6+4i)+(5+2i) & (-3i)+(-7+5i) \end{pmatrix} = \begin{pmatrix} 6 & 1-i \\ 11+6i & -7+2i \end{pmatrix}$$

$$(b) \quad CA = \begin{pmatrix} (1+2i)(4+i)+(i)(6+4i) & (1+2i)(-2+3i)+(i)(-3i) \\ (3-i)(4+i)+(8)(6+4i) & (3-i)(-2+3i)+(8)(-3i) \\ (4+2i)(4+i)+(1-i)(6+4i) & (4+2i)(-2+3i)+(1-i)(-3i) \end{pmatrix} = \begin{pmatrix} -2+15i & -5-i \\ 61+31i & -3-13i \\ 24+10i & -17+5i \end{pmatrix}$$

$$(c) \quad (2+i)B = \begin{pmatrix} (2+i)(2-i) & (2+i)(3-4i) \\ (2+i)(5+2i) & (2+i)(-7+5i) \end{pmatrix} = \begin{pmatrix} 5 & 10-5i \\ 8+9i & -19+3i \end{pmatrix}$$